INSTRUCTIONS TO STUDENTS

1. Please write your student number only. Do not write your name.
2. This assessment paper contains 5 questions and comprises 3 printed pages.
3. Students are required to answer ALL the questions.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. Each question carries 20 marks.
1. For each of the following multiple choice questions, indicate one only from among A to E as the best answer.
   i. In a classical micro-canonical ensemble the phase space distribution function
      A. $\rho$ is a constant on a hypersurface $H(p, q) = E$.
      B. $\rho$ is a constant in the region $H(p, q) < E$.
      C. $\rho$ is proportional to energy $E$.
      D. $\rho$ is proportional to $\exp(\beta H)$.
      E. $\rho$ is proportional to $\delta(H(p, q) - E)$.
   
   ii. The Jarzynski relation connecting work $W$ with Helmholtz free energy $F$ is
      A. $\langle e^{-\beta W} \rangle > e^{-\beta \Delta F}$.
      B. $\langle e^{-\beta W} \rangle < e^{-\beta \Delta F}$.
      C. $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$.
      D. $\langle e^{-\beta W} \rangle = e^{-\beta F}$.
      E. $\langle e^{\beta W} \rangle = e^{\beta \Delta F}$.
   
   iii. With the scaling hypothesis, $f(b^Y t, b^X h) = b^D f(t, h)$, for the singular part of the free energy per degree of freedom near critical point, which one of the following statements is not true or not implied?
      A. $D$ is the dimension of the system.
      B. $t = T - T_c$.
      C. Widom’s scaling law, $\beta(\delta - 1) = \gamma$.
      D. Fisher’s scaling law, $\gamma = (2 - \eta)\nu$.
      E. Rushbrooke’s scaling law, $\alpha + 2\beta + \gamma = 2$.
   
   iv. In a quantum ideal Bose gas, the critical Bose-Einstein condensation temperature $T_0$ at which the occupation number of the ground state acquired an average value $\langle n_0 \rangle$ of order $N$ (the number of particles of the system) can be estimated approximately by
      A. $k_B T_0 \approx \hbar \omega$.
      B. $k_B T_0 \approx \hbar^2 / \left(2m \left(\frac{V}{N}\right)^{\frac{3}{2}}\right)$.
      C. $k_B T_0 \approx \hbar^2 / (2mV^{\frac{3}{2}})$.
      D. $k_B T_0 \approx \langle m v^2 \rangle$.
      E. $k_B T_0 \approx 8a / (27b)$.
   
   v. Einstein equation is
      A. $E = mc^2$.
      B. $\langle x^2 \rangle = 2Dt$.
      C. $D = k_B T \/(6\pi \eta a)$.
      D. $D = \mu k_B T$.
      E. All of the above.
2. The eigen-energies of a single quantum particle in a one-dimensional box is \( E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2} \), where \( n = 1, 2, 3, \ldots \), and \( m \) is mass, \( L \) is the length of the interval where the particle is confined. Working in canonical ensemble:
   
   i. Determine the force that the particle exerts to the wall of the box, in the high-temperature limit, i.e., \( \beta = 1/(k_B T) \) is small.
   
   ii. Determine the same force, but in the opposite limit, i.e., temperature \( T \) is small. The answer should be accurate to the first order of a suitable small low-temperature expansion parameter.

Define \( \alpha = \frac{\pi^2 \hbar^2}{2m} \), then \( E_n = \alpha n^2/L^2 \), the partition function is \( Z = \sum_{n=1}^{\infty} e^{-\beta \alpha n^2/L^2} = x + x^4 + x^9 + \cdots \), where \( x = e^{-\alpha \beta /L^2} \). For low temperatures, \( \beta = 1/(k_B T) \) is large, and \( x \) is small, we just keep the first two terms in the Taylor expansion of \( x \) as a good approximation. For higher temperature, the summand varies slowly with \( n \) we approximate the sum by integral, \( Z \approx \int_0^{\infty} e^{-\alpha \beta n^2/L^2} \, dn = L \sqrt{2\pi m k_B T} / \hbar \), (exactly the same result as for in classical particle, i.e., \( \iint \frac{dx dp}{h} e^{-\frac{\beta p^2}{2m}} \) and the free energy is \( F = -k_B T \ln Z \).

   Using force \( f = -\frac{\partial F}{\partial L} \) we obtain, for i) high temperature \( f = k_B T / L \) (ideal gas law), ii) for low temperature \( f = \frac{2\alpha}{L^3} (1 + 3x^3 + \cdots) \).

3. For exact results in two-dimensional Ising models, duality plays an important role.
   
   i. Explain the meaning of duality in a topological sense and demonstrate that square lattice is self-dual.
   
   ii. State the duality relation, which relates high-temperature expansion with low-temperature expansion.
   
   iii. Give the duality argument that the critical temperature of the two-dimensional Ising model on a square lattice with nearest neighbor interaction is determined by \( \tanh K_c = \exp(-2K_c) \), where \( K_c = J/(k_B T_c) \).

   Exactly the same as in my notes, or at the elearning-week recordings.

4. Consider the Langevin equation in the case where the acceleration term \( mdv/dt \) is negligible, \( -k \frac{dx}{dt} + R(t) = 0 \), with a standard white noise for the random force, i.e., \( \langle R(t) \rangle = 0, \langle R(t) R(t') \rangle = C \delta(t - t') \).
i. Solve the stochastic differential equation assuming the initial condition $x(0) = 0$; determine the mean-square displacement $\langle x(t)^2 \rangle$ as a function of time $t$.

ii. Derive the Fokker-Planck equation for the probability density $\langle P(x, t) \rangle$ for the random variable $x$.

i) Integrate, we get $x(t) = \frac{1}{k} \int_0^t R(t')dt'$. Perform a two-dimensional integral over a square of $[0, t]^2$ and use the delta function correlation, we obtain, $\langle x(t)^2 \rangle = 1/k^2 \int_0^t dt_1 \int_0^t dt_2 \langle R(t_1)R(t_2) \rangle = Ct/k^2$.

ii) Probability conservation means $\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} (xp) = 0$. Using the expression for the rate of $x$, and moving the second term to the right and solving the equation formally, then taken average over noise, or alternatively, using analogy to the standard Fokker-Planck equation, we obtain (skipping steps), $\frac{\partial \langle P(x,t) \rangle}{\partial t} = \frac{C}{2k^2} \frac{\partial^2}{\partial x^2} \langle P(x,t) \rangle$, which is a diffusion equation.

5. Consider a simplified Boltzmann equation of the form $\frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla r f = -\frac{f - f_{eq}}{\tau}$, where $f_{eq}$ is local equilibrium distribution having the property $\langle \ln f_{eq} \rangle_f = \langle \ln f_{eq} \rangle_{f_{eq}}$. $m$ is mass of the particle, $\tau > 0$ is relaxation time constant, and the distribution function $f = f(r, p, t)$ is a function of (three-dimensional) position $r$, momentum $p$, and time $t$. The Boltzmann $H$-function is defined as $H = \int \int dr dp f \ln f$. Prove the $H$-theorem: $\frac{dH}{dt} \leq 0$.

The key steps are: $\frac{dH}{dt} = \int dr \int dp \frac{\partial f}{\partial t} (1 + \ln f)$. This is because, $t, r, p$ are independent variables. Use Boltzmann equation $\frac{\partial f}{\partial t} = -\frac{p}{m} \cdot \nabla r f - \frac{f - f_{eq}}{\tau}$, the first momentum term can be dropped, because, it can be written as $\int dr \nabla r (f \ln f)$ and can be changed to a surface integral using Gauss theorem. The $1$ can also be dropped because particle number conservation, $\int dr \int dp f = N$. We are left with $\frac{dH}{dt} = -\frac{1}{\tau} \int dr \int dp (f - f_{eq}) \ln f$. To proceed further, we use the given local equilibrium condition, which means $\int dr \int dp (f - f_{eq}) \ln f_{eq} = 0$. Dividing by $\tau$, adding into $dH/dt$, we get $\frac{dH}{dt} = -\frac{1}{\tau} \int dr \int dp (f - f_{eq}) \ln \frac{f}{f_{eq}} \leq 0$. This problem is an example in the book of R. Zwanzig, page 93-96.