Series solutions of ordinary differential equations

Second-order linear ordinary differential equations

Any homogeneous second-order linear ODE can be written in the form

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

where $y' = dy/dx$ and $p(x)$ and $q(x)$ are given functions of x. The most general solution to Eq. (1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \tag{2}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of Eq. (1), and c_1 and c_2 are constants. Their linear independence may be verified by the evaluation of the Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'. \tag{3}$$

If $W(x) \neq 0$ in a given interval, then y_1 and y_2 are linearly independent in that interval.

By differentiating Eq. (3) with respect to x, we obtain

$$W' = y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1' = y_1 y_2'' - y_1'' y_2.$$

Since both y_1 and y_2 satisfy Eq. (1), we may substitute for y_1'' and y_2'', which yields

$$\begin{aligned} W' &= -y_1(py_2' + qy_2) + (py_1' + qy_1)y_2 \\ &= -p(y_1 y_2' - y_1' y_2) = -pW. \end{aligned}$$

Integrating, we find

$$W(x) = C \exp\left\{-\int^x p(u)\,du\right\}, \qquad (4)$$

where C is a constant. If $p(x) \equiv 0$, we obtain $W =$ constant.

Ordinary and singular points of an ODE

Consider the second-order linear homogeneous ODE

$$y'' + p(z)y' + q(z) = 0, \tag{5}$$

where the functions are complex functions of a complex variable z. If at some point $z = z_0$ the functions $p(z)$ and $q(z)$ are finite, and can be expressed as complex power series

$$p(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n, \quad q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n,$$

then $p(z)$ and $q(z)$ are said to be analytic at $z = z_0$, and this point is called an ordinary point of the ODE. If, however, $p(z)$ or $q(z)$, or both, diverge at $z = z_0$, then it is called a singular point of the ODE.

Even if an ODE is singular at a given point $z = z_0$, it may still possess a non-singular (finite) solution at that point. The necessary and sufficient condition for such a solution to exist is that $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ are both analytic at $z = z_0$. Singular points that have this property are regular singular points, whereas any singular point not satisfying both these criteria is termed an irregular or essential singularity.

Example

Legendre's equation has the form

$$(1 - z^2)y'' - 2zy' + l(l+1)y = 0. \tag{6}$$

where l is a constant. Show that $z = 0$ is an ordinary point and $z = \pm 1$ are regular singular points of this equation.

Answer

We first divide through by $1 - z^2$ to put the equation into our standard form, Eq. (5):

$$y'' - \frac{2z}{1 - z^2}y' + \frac{l(l+1)}{1 - z^2}y = 0.$$

Comparing with Eq. (5), we identify

$$\begin{aligned} p(z) &= \frac{-2z}{1 - z^2} = \frac{-2z}{(1+z)(1-z)} \\ q(z) &= \frac{l(l+1)}{1 - z^2} = \frac{l(l+1)}{(1+z)(1-z)}. \end{aligned}$$

By inspection, $p(z)$ and $q(z)$ are analytic at $z = 0$, which is therefore an ordinary point, but both diverge for $z = \pm 1$, which are thus singular points. However, at $z = 1$, we see that both $(z-1)p(z)$ and $(z-1)^2 q(z)$ are analytic, and hence $z = 1$ is a regular singular point. Similarly, $z = -1$ is a regular singular point.

Example

Show that the Legendre's equation has a regular singularity at $|z| \to \infty$.

Answer

Letting $w = 1/z$, the derivatives with respect to z become

$$\begin{aligned}
\frac{dy}{dz} &= \frac{dy}{dw}\frac{dw}{dz} = -\frac{1}{z^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}, \\
\frac{d^2y}{dz^2} &= \frac{dw}{dz}\frac{d}{dw}\left(\frac{dy}{dz}\right) \\
&= -w^2\left(-2w\frac{dy}{dw} - w^2\frac{d^2y}{dw^2}\right) \\
&= w^3\left(2\frac{dy}{dw} + w\frac{d^2y}{dw^2}\right).
\end{aligned}$$

If we substitute these derivatives into Legendre's equation Eq. (6) we obtain

$$\left(1 - \frac{1}{w^2}\right)w^3\left(2\frac{dy}{dw} + w\frac{d^2y}{dw^2}\right)$$
$$+2\frac{1}{w}w^2\frac{dy}{dw} + l(l+1)y = 0,$$

which simplifies to give

$$w^2(w^2-1)\frac{d^y}{dw^2}+2w^3\frac{dy}{dw}+l(l+1)y=0.$$

Dividing through by $w^2(w^2-1)$, and comparing with Eq. (5), we identify

$$\begin{aligned} p(w) &= \frac{2w}{w^2-1}, \\ q(w) &= \frac{l(l+1)}{w^2(w^2-1)}. \end{aligned}$$

At $w=0$, $p(w)$ is analytic but $q(w)$ diverges, and so the point $|z|\to\infty$ is a singular point of Legendre's equation. However, since wp and w^2q are both analytic at $w=0$, $|z|\to\infty$ is a regular singular point.

Equation	Regular singularities	Essential singularities
Legendre		
$(1-z^2)y''$		
$-2zy' + l(l+1)y = 0$	$-1, 1, \infty$	
Chebyshev		
$(1-z^2)y''$		
$-zy' + n^2 y = 0$	$-1, 1, \infty$	
Bessel		
$z^2 y'' + zy'$		
$+(z^2 - \nu^2)y = 0$	0	∞
Laguerre		
$zy'' + (1-z)y'$		
$+\alpha y = 0$	0	∞
Simple harmonic oscillator		
$y'' + \omega^2 y = 0$		∞

Series solutions about an ordinary point

If $z = z_0$ is an ordinary point of Eq. (5), then every solution $y(z)$ of the equation is also analytic at $z = z_0$. We shall take z_0 as the origin. If this is not the case, then a substitution $Z = z - z_0$ will make it so. Then $y(z)$ can be written as

$$y(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{7}$$

Such a power series converges for $|z| < R$, where R is the radius of convergence.

Since every solution of Eq. (5) is analytic at an ordinary point, it is always possible to obtain two independent solutions of the form, Eq. (7).

The derivatives of y with respect to z are given by

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n, \quad (8) \\
y'' &= \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n. \quad (9)
\end{aligned}
$$

By substituting Eqs. (7)–(9) into the ODE, Eq. (5), and requiring that the coefficients of each power of z sum to zero, we obtain a recurrence relation expressing a_n as a function of the previous a_r $(0 \le r \le n-1)$.

Example

Find the series solutions, about $z = 0$, of

$$y'' + y = 0.$$

Answer

$z =$ is an ordinary point of the equation, and so we may obtain two independent solutions by making the substitution $y = \sum_{n=0}^{\infty} a_n z^n$. Using Eq. (7) and (9), we find

$$\sum_{n=0}^{\infty} (n+1)(n+1)a_{n+1}z^n + \sum_{n=0}^{\infty} a_n z^n = 0,$$

which may be written as

$$\sum_{n=0}^{\infty} [(n+1)(n+1)a_{n+2} + a_n]z^n = 0.$$

For this equation to be satisfied, we require that the coefficient of each power of z vanishes separately, and so we obtain the two-term recurrence relation

$$a_{n+1} = -\frac{a_n}{(n+2)(n+1)} \quad \text{for} \quad n \geq 0.$$

Two independent solutions of the ODE may be obtained by setting either $a_0 = 0$ or $a_1 = 0$. If we first set $a_1 = 0$ and choose $a_0 = 1$, then we obtain the solution

$$y_1(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

However, if we set $a_0 = 0$ and choose $a_1 = 1$, we obtain a second independent solution

$$y_2(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

Recognizing these two series as $\cos z$ and $\sin z$, we can write the general solution as

$$y(z) = c_1 \cos z + c_2 \sin z,$$

where c_1 and c_2 are arbitrary constants.

Example

Find the series solutions, about $z = 0$, of

$$y'' - \frac{2}{(1-z)^2} y = 0.$$

Answer

$z = 0$ is an ordinary point, and we may therefore find two independent solutions by substituting $y = \sum_{n=0}^{\infty} a_n z^n$. Using Eq. (8) and (9), and multiplying through by $(1-z)^2$, we find

$$(1 - 2z + z^2) \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} a_n z^n = 0,$$

which leads to

$$\sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} n(n-1) a_n z^{n-1}$$
$$+ \sum_{n=0}^{\infty} n(n-1) a_n z^n - 2 \sum_{n=0}^{\infty} a_n z^n = 0.$$

In order to write all these series in terms of the coefficients of z^n, we must shift the summation index in the first two sums to obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n - 2\sum_{n=0}^{\infty}(n+1)na_{n+1}z^n + \sum_{n=0}^{\infty}(n^2-n-2)a_n z^n = 0,$$

which can be written as

$$\sum_{n=0}^{\infty}(n+1)[(n+2)a_{n+2}-2na_{n+1}+(n-2)a_n]z^n = 0.$$

By demanding that the coefficients of each power of z vanish separately, we obtain the recurrence relation

$$(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n = 0 \quad \text{for } n \geq 0,$$

which determines a_n for $n \geq 2$ in terms of a_0 and a_1.

One solution has $a_n = a_0$ for all n, in which case (choosing $a_0 = 1$), we find

$$y_1(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}.$$

The other solution to the recurrence relation is $a_1 = -2a_0$, $a_2 = a_0$ and $a_n = 0$ for $n > 2$, so that we obtain a polynomial solution to the ODE:

$$y_2(z) = 1 - 2z + z^2 = (1-z)^2.$$

The linear independence of y_1 and y_2 can be checked:

$$\begin{aligned} W &= y_1 y_2' - y_1' y_2 \\ &= \frac{1}{1-z}[-2(1-z)] - \frac{1}{(1-z)^2}(1-z)^2 = -3. \end{aligned}$$

The general solution of the ODE is therefore

$$y(z) = \frac{c_1}{1-z} + c_2(1-z)^2.$$

Series solutions about a regular singular point

If $z = 0$ is a regular singular point of the equation

$$y'' + p(z)y' + q(z)y = 0,$$

then $p(z)$ and $q(z)$ are not analytic at $z = 0$. But there exists at least one solution to the above equation, of the form

$$y = z^\sigma \sum_{n=0}^{\infty} a_n z^n, \tag{10}$$

where the exponent σ may be real or complex number, and where $a_0 \neq 0$. Such a series is called a generalized power series or Frobenius series.

Since $z = 0$ is a regular singularity of the ODE, $zp(z)$ and $z^2 q(z)$ are analytic at $z = 0$, so that we may write

$$\begin{aligned} zp(z) &= s(z) = \sum_{n=0}^{\infty} s_n z^n \\ z^2 q(z) &= t(z) = \sum_{n=0}^{\infty} t_n z^n. \end{aligned}$$

The original ODE therefore becomes

$$y'' + \frac{s(z)}{z} y' + \frac{t(z)}{z^2} y = 0.$$

Let us substitute the Frobenius series Eq. (10) into this equation. The derivatives of Eq. (10) with respect to x are given by

$$y' = \sum_{n=0}^{\infty} (n + \sigma) a_n z^{n+\sigma-1} \tag{11}$$

$$y'' = \sum_{n=0}^{\infty} (n + \sigma)(n + \sigma - 1) a_n z^{n+\sigma-2}, \tag{12}$$

and we obtain

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2}$$

$$+s(z)\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-2}+t(z)\sum_{n=0}^{\infty}a_n z^{n+\sigma-2}=0.$$

Dividing this equation through by $z^{\sigma-2}$ we find

$$\sum_{n=0}^{\infty}[(n+\sigma)(n+\sigma-1)+s(z)(n+\sigma)+t(z)]a_n z^n=0. \tag{13}$$

Setting $z=0$, all terms in the sum with $n>0$ vanish, so that

$$[\sigma(\sigma-1)+s(0)\sigma+t(0)]a_0=0,$$

which, since we require $a_0\neq 0$, yields the indicial equation

$$\sigma(\sigma-1)+s(0)\sigma+t(0)=0. \tag{14}$$

This equation is a quadratic in σ and in general has two roots. The two roots of the indicial equation σ_1 and σ_2 are called the indices of the regular singular point. By substituting each of these roots into Eq. (13) in turn, and requiring that the coefficients of each power of z vanish separately, we obtain a recurrence relation (for each root) expressing each a_n as a function of the previous a_r $(0 \leq r \leq n-1)$. Depending on the roots of the indicial equation σ_1 and σ_2, there are three possible general cases.

Distinct roots not differing by an integer

If the roots of the indicial equation σ_1 and σ_2 differ by an amount that is not an integer, then the recurrence relations corresponding to each root lead to two linearly independent solutions of the ODE,

$$y_1(z) = z^{\sigma_1} \sum_{n=0}^{\infty} a_n z^n, \quad y_2(z) = z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n.$$

The linear independence of these two solutions follows from the fact that y_2/y_1 is not a constant since $\sigma_1 - \sigma_2$ is not an integer. Since y_1 and y_2 are linearly independent we may use them to construct the general solution $y = c_1 y_1 + c_2 y_2$.

We also note that this case includes complex conjugate roots where $\sigma_2 = \sigma_1^*$, since $\sigma_1 - \sigma_2 = \sigma_1 - \sigma_1^* = 2i \,\mathrm{Im}\, \sigma_1$ cannot be equal to a real integer.

Example

Find the power series solutions about $z = 0$ of

$$4zy'' + 2y' + y = 0.$$

Answer

Dividing through by $4z$, we obtain

$$y'' + \frac{1}{2z}y' + \frac{1}{4z}y = 0, \qquad (15)$$

and on comparing with Eq. (5) we identify $p(z) = 1/(2z)$ and $q(z) = 1/(4z)$. Clearly $z = 0$ is a singular point of Eq. (15), but since $zp(z) = 1/2$ and $z^2 q(z) = z/4$ are finite there, it is a regular singular point. We therefore substitute the Frobenius series $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ into Eq. (15). Using Eq. (11) and (12), we obtain

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2}$$

$$+\frac{1}{2z}\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} + \frac{1}{4z}\sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0,$$

which on dividing through by $z^{\sigma-2}$ gives

$$\sum_{n=0}^{\infty}[(n+\sigma)(n+\sigma-1)+\frac{1}{2}(n+\sigma)+\frac{1}{4}z]a_n z^n = 0. \tag{16}$$

If we set $z = 0$ then all terms in the sum with $n > 0$ vanish, and we obtain the indicial equation

$$\sigma(\sigma-1)+\frac{1}{2}\sigma = 0,$$

which has roots $\sigma = 1/2$ and $\sigma = 0$. Since these roots do not differ by an integer, we expect to find two independent solutions to Eq. (15), in the form of Frobenius series.

Demanding that the coefficients of z^n vanish separately in Eq. (16), we obtain the recurrence relation

$$(n+\sigma)(n+\sigma-1)a_n+\frac{1}{2}(n+\sigma)a_n+\frac{1}{4}a_{n-1} = 0. \tag{17}$$

If we choose the larger root of the indicial equation, $\sigma = 1/2$, this becomes

$$(4n^2 + 2n)a_n + a_{n-1} = 0 \Rightarrow a_n = \frac{-a_{n-1}}{2n(2n+1)}.$$

Setting $a_0 = 1$ we find $a_n = (-1)^n/(2n+1)!$ and so the solution to Eq. (15) is

$$\begin{aligned} y_1(z) &= \sqrt{z}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^n \\ &= \sqrt{z} - \frac{(\sqrt{z})^3}{3!} + \frac{(\sqrt{z})^5}{5!} - \cdots = \sin\sqrt{z}. \end{aligned}$$

To obtain the second solution, we set $\sigma = 0$ in Eq. (17), which gives

$$(4n^2 - 2n)a_n + a_{n-1} = 0 \Rightarrow a_n = -\frac{a_{n-1}}{2n(2n-1)}.$$

Setting $a_0 = 1$ now gives $a_n = (-1)^n/(2n)!$, and so the second solution to Eq. (15) is

$$\begin{aligned} y_2(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^n \\ &= 1 - \frac{(\sqrt{z})^2}{2!} + \frac{(\sqrt{z})^4}{4!} - \cdots = \cos\sqrt{z}. \end{aligned}$$

We may check that $y_1(z)$ and $y_2(z)$ are linearly independent by computing the Wronskian

$$
\begin{aligned}
W &= y_1 y_2' - y_2 y_1' \\
&= \sin\sqrt{z}\left(-\frac{1}{2\sqrt{z}}\sin\sqrt{z}\right) \\
&\quad -\cos\sqrt{z}\left(\frac{1}{2\sqrt{z}}\cos\sqrt{z}\right) \\
&= -\frac{1}{2\sqrt{z}}(\sin^2\sqrt{z}+\cos^2\sqrt{z}) \\
&= -\frac{1}{2\sqrt{z}} \neq 0.
\end{aligned}
$$

Since $W \neq 0$ the solutions $y - 1(z)$ and $y_2(z)$ are linearly independent. Hence the general solution to Eq. (15) is given by

$$y(z) = c_1 \sin\sqrt{z} + c_2 \cos\sqrt{z}.$$

Repeated root of the indicial equation

If the indicial equation has a repeated root, so that $\sigma_1 = \sigma_2 = \sigma$, then only one solution in the form of a Frobenius series Eq. (10) may be found

$$y_1(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n.$$

Distinct roots differing by an integer

If the roots of the indicial equation differ by an integer, then the recurrence relation corresponding to the larger of the two roots leads to a solution of the ODE. For complex roots, the ‘larger’ root is taken to be the one with the larger real part.

Example

Find the power series solutions about $z = 0$ of

$$z(z-1)y'' + 3zy' + y = 0. \tag{18}$$

Answer

Dividing through by $z(z-1)$, we put

$$y'' + \frac{3}{(z-1)}y' + \frac{1}{z(z-1)}y = 0, \tag{19}$$

and on comparing with Eq. (5), we identify $p(z) = 3/(z-1)$ and $q(z) = 1/[z(z-1)]$. We see that $z = 0$ is a singular point of Eq. (19), but since $zp(z) = 3z/(z-1)$ and $z^2q(z) = z/(z-1)$ are finite there, it is a regular singular point. We substitute $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ into Eq. (19), and using Eq. (11) and (12), we obtain

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2}$$

$$+\frac{3}{z-1}\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1}$$

$$+\frac{1}{z(z-1)}\sum_{n=0}^{\infty}a_n z^{n+\sigma}=0,$$

which on dividing through by $z^{\sigma-2}$ gives

$$\sum_{n=0}^{\infty}\Bigg[(n+\sigma)(n+\sigma-1)+\frac{3z}{z-1}(n+\sigma)$$

$$+\frac{z}{z-1}\Bigg]a_n z^n=0.$$

Multiplying through by $z-1$, we obtain

$$\sum_{n=0}^{\infty}[(z-1)(n+\sigma)(n+\sigma-1)+3z(n+\sigma)+z]a_n z^n=0. \tag{20}$$

If we set $z = 0$, then all terms in the sum with $n > 0$ vanish, and we obtain the indicial eqaution

$$\sigma(\sigma - 1) = 0,$$

which has the roots $\sigma = 1$ and $\sigma = 0$.

Since the roots differ by an integer, it may not be possible to find two linearly independent solutions of Eq. (19) in the form of Frobenius series. We are guaranteed to find one such solution corresponding to the larger root, $\sigma = 1$.

Demanding that the coefficients of z^n vanish separately in Eq. (20), we obtain the recurrence relation

$$\begin{aligned}(n-1+\sigma)(n-2+\sigma)a_{n-1} - (n+\sigma)(n+\sigma-1)a_n \\ +3(n-1+\sigma)a_{n-1} + a_{n-1} = 0,\end{aligned}$$

which may be simplified to give

$$(n+\sigma-1)a_n = (n+\sigma)a_{n-1}. \qquad (21)$$

Substituting $\sigma = 1$ into this expression, we obtain

$$a_n = \left(\frac{n+1}{n}\right) a_{n-1},$$

and setting $a_0 = 1$ we find $a_n = n + 1$, so one solution to Eq. (19) is

$$\begin{aligned} y_1(z) &= z\sum_{n=0}^{\infty}(n+1)z_n = z(1 + 2z + 3z^2 + \cdots) \\ &= \frac{z}{(1-z)^2}. \end{aligned} \tag{22}$$

If we attempt to find a second solution (corresponding to the smaller root of the indicial equation) by setting $\sigma = 0$ in Eq. (21), we find

$$a_n = \left(\frac{n}{n-1}\right) a_{n-1},$$

but we require $a_0 \neq 0$, so a_1 is formally infinite and the method fails.

Obtaining a second solution

The Wronskian method

If y_1 and y_2 are linearly independent solutions of the standard equation

$$y'' + p(z)y' + q(z)y = 0,$$

then the Wronskian of these two solutions is given by $W(x) = y_1 y_2' - y_2 y_1'$. Dividing the Wronskian by y_1^2 we obtain

$$\frac{W}{y_1^2} = \frac{y_2'}{y_1} - \frac{y_1'}{y_1^2} y_2$$

$$= \frac{y_2'}{y_1} + \left[\frac{d}{dz}\left(\frac{1}{y_1}\right)\right] y_2 = \frac{d}{dz}\left(\frac{y_2}{y_1}\right),$$

which integrates to give

$$y_2(x) = y_1(x) \int^x \frac{W(u)}{y_1^2(u)} du.$$

Using the alternative expression for $W(z)$ given in Eq. (4) with $C = 1$, we find

$$y_2(z) = y_1(z) \int^z \frac{1}{y_1^2(u)} \exp\left\{ - \int^u p(v)\, dv \right\} du. \tag{23}$$

Hence, given y_1, we can calculate y_2.

Example

Find a second solution to Eq. (19) using the Wronskian method.

Answer

For the ODE Eq. (19) we have $p(z) = 3/(z-1)$, and from Eq. (22) we see that one solution to Eq. (19) is $y_1 = z/(1-z)^2$. Substituting for p and y_1 in Eq. (23) we have

$$\begin{aligned} y_2(z) &= \frac{z}{(1-z)^2} \int^z \frac{(1-u)^4}{u^2} \exp\left(-\int^u \frac{3}{v-1} dv\right) du \\ &= \frac{z}{(1-z)^2} \int^z \frac{(1-u)^4}{u^2} \exp[-3\ln(u-1)]\, du \\ &= \frac{z}{(1-z)^2} \int^z \frac{u-1}{u^2}\, du \\ &= \frac{z}{(1-z)^2} \left(\ln z + \frac{1}{z}\right). \end{aligned}$$

The derivative method

The method begins with the derivation of a recurrence relation for the coefficients a_n in the Frobenius series solution. However, rather than putting $\sigma = \sigma_1$ in this recurrence relation to evaluate the first series solution, we instead keep σ as a variable parameter. This means that the computed a_n are functions of σ and the computed solution is now a function of z and σ

$$y(z, \sigma) = z^{\sigma} \sum_{n=0}^{\infty} a_n(\sigma) z^n. \tag{24}$$

Let

$$\mathcal{L} = \frac{d^2}{dx^2} + p(z)\frac{d}{dx} + q(z),$$

The series $\mathcal{L}y(z, \sigma)$ will contain only a term in z^{σ}, since the recurrence relation defining the $a_n(\sigma)$ is such that these coefficients vanish for higher powers of z. But the coefficients of z^{σ} is simply the LHS of the indicial equation.

Therefore, if the roots of the indicial equation are $\sigma = \sigma_1$ and $\sigma = \sigma_2$, it follows that

$$\mathcal{L}y(z, \sigma) = a_0(\sigma - \sigma_1)(\sigma - \sigma_2)z^\sigma. \qquad (25)$$

Thus, for $y(z, \sigma)$ to be a solution of the ODE $\mathcal{L}y = 0$, σ must equal σ_1 or σ_2. For simplicity, we shall set $a_0 = 1$ in the following discussion.

Let us first consider the case in which the two roots of the indicial equation are equal, $\sigma_1 = \sigma_2$. From Eq. (25) we then have

$$\mathcal{L}y(z, \sigma) = (\sigma - \sigma_1)^2 z^n.$$

Differentiating this equation with respect to σ, we obtain

$$\frac{\partial}{\partial \sigma}[\mathcal{L}y(z, \sigma)] = (\sigma - \sigma_1)^2 z^\sigma \ln z + 2(\sigma - \sigma_1)z^\sigma,$$

which equal zero if $\sigma = \sigma_1$. But since $\partial/\partial\sigma$ and $\mathcal{L}$ are operators that differentiate with respect to different variables, we can reverse their order, so that

$$\mathcal{L}\left[\frac{\partial}{\partial \sigma} y(z, \sigma)\right] = 0 \text{ at } \sigma = \sigma_1.$$

Hence the function in square brackets, evaluated at $\sigma = \sigma_1$, and denoted by

$$\left[\frac{\partial}{\partial\sigma} y(z,\sigma)\right]_{\sigma=\sigma_1}, \tag{26}$$

is also a solution of the original ODE $\mathcal{L}y = 0$, and is the second linearly independent solution.

The case by which the roots of the indicial equation differ by an integer is now treated. In Eq. (25), since $\mathcal{L}$ differentiates with respect to z we may multiply Eq. (25) by any function of σ, say $\sigma - \sigma_2$, and take this function inside $\mathcal{L}$ on the LHS to obtain

$$\mathcal{L}[(\sigma - \sigma_2)y(z,\sigma)] = (\sigma - \sigma_1)(\sigma - \sigma_2)^2 z^\sigma. \tag{27}$$

Therefore the function

$$[(\sigma - \sigma_2)y(z,\sigma)]_{\sigma=\sigma_2}$$

is also a solution of the ODE $\mathcal{L}y = 0$.

Since this function is not linearly independent with $y(z, \sigma_1)$, we must find another solution. Differentiating Eq. (27) with respect to σ, we find

$$\frac{\partial}{\partial\sigma}\{\mathcal{L}[(\sigma-\sigma_2)y(z,\sigma)]\} = (\sigma-\sigma_2)^2 z^\sigma$$
$$+2(\sigma-\sigma_1)(\sigma-\sigma_2)z^\sigma + (\sigma-\sigma_1)(\sigma-\sigma_2)^2 z^\sigma \ln z,$$

which is equal to zero if $\sigma = \sigma_2$. Therefore,

$$\mathcal{L}\left\{\frac{\partial}{\partial\sigma}[(\sigma-\sigma_2)y(z,\sigma)]\right\} = 0 \text{ at } \sigma = \sigma_2,$$

and so the function

$$\left\{\frac{\partial}{\partial\sigma}[(\sigma-\sigma_2)y(z,\sigma)]\right\}_{\sigma=\sigma_2} \qquad (28)$$

is also a solution of the original ODE $\mathcal{L}y = 0$, and is the second linearly independent solution.

Example

Find a second solution to Eq. (19) using the derivative method.

Answer

From Eq. (21) the recurrence relation (with σ as a parameter) id given by

$$(n+\sigma-1)a_n = (n+\sigma)a_{n-1}.$$

Setting $a_0 = 1$ we find that the coefficients have the particularly simple form $a_n(\sigma) = (\sigma+n)/\sigma$. We therefore consider the function

$$y(z,\sigma) = z^\sigma \sum_{n=0}^{\infty} a_n(\sigma) z^n = z^\sigma \sum_{n=0}^{\infty} \frac{\sigma+n}{\sigma} z^n.$$

The smaller toot of the indicial equation for Eq. (19) is $\sigma_2 = 0$, and so from Eq. (28) a second linearly independent solution to the ODE is

$$\left\{\frac{\partial}{\partial\sigma}[\sigma y(z,\sigma)]\right\}_{\sigma=0} = \left\{\frac{\partial}{\partial\sigma}\left[z^\sigma \sum_{n-0}^{\infty}(\sigma+n)z^n\right]\right\}_{\sigma=0}$$

The derivative with respect to σ is given by

$$\frac{\partial}{\partial\sigma}\left[z^\sigma\sum_{n=0}^{\infty}(\sigma+n)z^n\right] = z^\sigma \ln z \sum_{n=0}^{\infty}(\sigma+n)z^n + z^\sigma \sum_{n=0}^{\infty} z^n,$$

which on setting $\sigma = 0$ gives the second solution

$$\begin{aligned} y_2(z) &= \ln z \sum_{n=0}^{\infty} n z^n + \sum_{n=0}^{\infty} z^n \\ &= \frac{z}{(1-z)^2}\ln z + \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\left(\ln z + \frac{1}{z} - 1\right). \end{aligned}$$

Series form of the second solution

Let us consider the case when the two solutions of the indicial equation are equal. In this case, a second solution is given by Eq. (26),

$$
\begin{aligned}
y_2(z) &= \left[\frac{\partial y(z,\sigma)}{\partial \sigma}\right]_{\sigma=\sigma_1} \\
&= (\ln z) z^{\sigma_1} \sum_{n=0}^{\infty} a_n(\sigma_1) z^n \\
&\quad + z^{\sigma_1} \sum_{n=0}^{\infty} \left[\frac{da_n(\sigma)}{d\sigma}\right]_{\sigma=\sigma_1} z^n \\
&= y_1(z) \ln z + z^{\sigma_1} \sum_{n=1}^{\infty} b_n z^n,
\end{aligned}
$$

where $b_n = [da_n(\sigma)/d\sigma]_{\sigma=\sigma_1}$.

In the case where the roots of the indicial equation differ by an integer, then from Eq. (28) a second solution is given by

$$\begin{aligned} y_2(z) &= \left\{ \frac{\partial}{\partial \sigma}[(\sigma - \sigma_2) y(z, \sigma)] \right\}_{\sigma = \sigma_2} \\ &= \ln z \left[(\sigma - \sigma_2) z^{\sigma} \sum_{n=0}^{\infty} a_n(\sigma) z^n \right]_{\sigma = \sigma_2} \\ &\quad + z^{\sigma_2} \sum_{n=0} \left[\frac{d}{d\sigma} (\sigma - \sigma_2) a_n(\sigma) \right]_{\sigma = \sigma_2} z^n. \end{aligned}$$

But $(\sigma - \sigma_2) y(z, \sigma)]$ at $\sigma = \sigma_2$ is just a multiple of the first solution $y(z, \sigma_1)$. Therefore the second solution is of the form

$$y_2(z) = c y_1(z) \ln z + z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n,$$

where c is a constant.

Polynomial solutions

Example

Find the power series solutions about $z = 0$ of

$$y'' - 2zy' + \lambda y = 0. \tag{29}$$

For what values of λ does the equation possess a polynomial solution? Find such a solution for $\lambda = 4$.

Answer

$z = 0$ is an ordinary point of Eq. (29) and so we look for solutions of the form $y = \sum_{n=0}^{\infty} z_n z^n$. Substituting this into the ODE and multiplying through by z^2, we find

$$\sum_{n=0}^{\infty} [n(n-1) - 2z^2 n + \lambda z^2] a_n z^n = 0.$$

By demanding that the coefficients of each power of z vanish separately we derive the recurrence relation

$$n(n-1)a_n - 2(n-2)a_{n-2} + \lambda a_{n-2} = 0,$$

which may be rearranged to give

$$a_n = \frac{2(n-2)-\lambda}{n(n-1)} a_{n-2} \quad \text{for } n \geq 2. \tag{30}$$

The odd and even coefficients are therefore independent of one another, and two solutions to Eq. (29) may be derived. We sither set $a_1 = 0$ and $a_0 = 1$ to obtain

$$y_1(z) = 1 - \lambda\frac{z^2}{2!} - \lambda(4-\lambda)\frac{z^4}{4!} - \lambda(4-\lambda)(8-\lambda)\frac{z^6}{6!} - \cdots, \tag{31}$$

or set $a_0 = 0$ and $a_1 = 1$ to obtain

$$\begin{aligned} y_2(z) \quad = \quad & z - (2-\lambda)\frac{z^3}{3!} - (2-\lambda)(6-\lambda)\frac{z^5}{5!} \\ & -(2-\lambda)(6-\lambda)(10-\lambda)\frac{z^7}{7!} - \cdots. \end{aligned}$$

Now from the recurrence relation Eq. (30) we see that for the ODE to possess a polynomial solution we require $\lambda = 2(n-2)$ for $n \geq 2$, or more simply $\lambda = 2n$ for $n \geq 0$, i.e. λ must be an even positive integer. If $\lambda = 4$ then from Eq. (31) the ODE has the polynomial solution

$$y_1(z) = 1 - \frac{4z^2}{2!} = 1 - 2z^2.$$

ALTERNATIVE SOLUTION:

We assume a polynomial solution to Eq. (29) of the form $y = \sum_{n=0}^{N} a_n z^n$. Substituting this form into Eq. (29) we find

$$\sum_{n=0}^{\infty} [n(n-1)a_n z^{n-2} - 2zna_n z^{n-1} + \lambda n a_n z^n] = 0.$$

Starting with the highest power of z, we demand that the coefficient of z^N vanishes. We require $-2N + \lambda = 0$ or $\lambda = 2N$.

Legendre's equation

Legendre's equation is

$$(1 - z^2)y'' - 2zy' + l(l + 1)y = 0. \qquad (32)$$

In normal usage, the variable z is the cosine of the polar angle in spherical polars, and thus $-1 \le z \le 1$. The parameter l is a given real number, and any solution of Eq. (32) is called a Legendre function.

$z = 0$ is an ordinary point of Eq. (32), and so there are two linearly independent solutions of the form $y = \sum_{n=0}^{\infty} a_n z^n$. Substituting we find

$$\sum_{n=0}^{\infty}[n(n-1)a_n z^{n-2} - n(n-1)a_n z^n$$
$$-2na_n z^n + l(l+1)a_n z^n] = 0,$$

which on collecting terms gives

$$\sum_{n=0}^{\infty}\{(n+2)(n+1)a_{n+2}$$
$$-[n(n+1) - l(l+1)]a_n\}z^n = 0.$$

The recurrence relation is therefore

$$a_{n+2} = \frac{[n(n+1) - l(l+1)]}{(n+1)(n+2)} a_n, \qquad (33)$$

for $n = 0, 1, 2, \ldots$. If we choose $a_0 = 1$ and $a_1 = 0$ then we obtain the solution

$$\begin{aligned} y_1(z) &= 1 - l(l+1)\frac{z^2}{2!} \\ &\quad + (l-2)l(l+1)(l+3)\frac{z^4}{4!} - \cdots, \end{aligned} \qquad (34)$$

whereas choosing $a_0 = 0$ and $a_1 = 1$, we find a second solution

$$\begin{aligned} y_2(z) &= z - (l-1)(l+2)\frac{z^3}{3!} \\ &\quad + (l-3)(l-1)(l+2)(l+4)\frac{z^5}{5!} - \cdots. \end{aligned} \qquad (35)$$

Both series converge for $|z| < 1$, so their radius of convergence is unity. Since Eq. (34) contains only even powers of z and (35) only odd powers, we can write the general solution to Eq. (32) as $y = c_1 y_1 + c_2 y_2$ for $|z| < 1$.

General solution for integer l

Now, if l is an integer in Eq. (32), i.e. $l = 0, 1, 2, \ldots$ then the recurrence relation Eq. (33) gives

$$a_{l+2} = \frac{[l(l+1) - l(l+1)]}{(l+1)(l+2)} a_l = 0,$$

so that the series terminates and we obtain a polynomial solution of order l. These solutions are called Legendre polynomial of order l; they are written $P_l(z)$ and are valid for all finite z. It is convenient to normalize $P_l(z)$ in such a way that $P_l(1) = 1$, and as a consequence $P_l(-1) = (-1)^l$. The first few Legendre polynomials are given by

$$P_0(z) = 1 \qquad P_1(z) = z$$

$$P_2(z) = \frac{1}{2}(3z^2 - 1) \qquad P_3(z) = \frac{1}{2}(5z^3 - 3z)$$

$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$$

$$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z).$$

According to whether l is an even or odd integer respectively, either $y_1(z)$ in Eq. (34) or $y_2(z)$ in Eq. (35) terminates to give a multiple of the corresponding Legendre polynomial $P_l(z)$. In either case, however, the other series does not terminate and therefore converges only for $|z| < 1$. According to whether l is even or odd we define Legendre function of the second kind as $Q_l(z) = \alpha_l y_2(z)$ or $Q_l(z) = \beta_l y_1(z)$ respectively, where the constants α_l and β_l are conventionally taken to have the values

$$\alpha_l = \frac{(-1)^{l/2} 2^l [(l/2)!]^2}{l!} \quad \text{for } l \text{ even,} \tag{36}$$

$$\beta_l = \frac{(-1)^{(l+1)/2} 2^{l-1} \{[(l-1)/2]!\}^2}{l!} \quad \text{for } l \text{ odd.} \tag{37}$$

These normalization factors are chosen so that the $Q_l(z)$ obey the same recurrence relations as the $P_l(z)$. The general solution of Legendre's equation for integer l is therefore

$$y(z) = c_1 P_l(z) + c_2 Q_l(z), \tag{38}$$

where $P_l(z)$ is a polynomial of order l and so converges for all z, and $Q_l(z)$ is an infinite series that converges only for $|z| < 1$.

Example

Use the Wronskian method to find a closed-form expression for $Q_0(z)$.

Answer

From Eq. (23) a second solution to Legendre's equation Eq. (32), with $l = 0$, is

$$
\begin{aligned}
y_2(z) &= P_0(z) \int^z \frac{1}{[P_0(u)]^2} \exp\left(\int^u \frac{2v}{1-v^2} dv \right) du \\
&= \int^z \exp\left[-\ln(1-u^2) \right] du \\
&= \int^z \frac{du}{(1-u^2)} = \frac{1}{2} \ln\left(\frac{1+z}{1-z} \right). \qquad (39)
\end{aligned}
$$

Expanding the logarithm in Eq. (39) as a Maclaurin series we obtain

$$y_2(z) = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots.$$

Comparing this with the expression for $Q_0(z)$, using Eq. (35) with $l = 0$ and the normalization Eq. (36), we find that $y_2(z)$ is already correctly normalized, and so

$$Q_0(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right).$$

Properties of Legendre polynomials

Rodrigues' formula

Rodrigues' formula for the $P_l(z)$ is

$$P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l. \tag{40}$$

To prove that this is a representation, we let $u = (z^2-1)^l$, so that $u' = 2lz(z^2-1)^{l-1}$ and

$$(z^2-1)u' - 2lzu = 0.$$

If we differentiate this expression $l+1$ times using Leibnitz' theorem, we obtain

$$\left[(z^2-1)u^{(l+2)} + 2z(l+1)u^{(l+1)} + l(l+1)u^{(l)}\right] \\ -2l\left[zu^{(l+1)} + (l+1)u^{(l)}\right] = 0,$$

which reduces to

$$(z^2-1)u^{(l+2)} + 2zu^{(l+1)} - l(l+1)u^{(l)} = 0.$$

Changing the sign all through, and comparing the resulting expression with Legendre's equation Eq. (32), we see that $u^{(l)}$ satisfies the same equation as $P_l(z)$, so that

$$u^{(l)}(z) = c_l P_l(z), \tag{41}$$

for some constant c_l that depends on l. To establish the value of c_l, we note that the only term in the expression for the lth derivative of $(z^2 - 1)^l$ that does not contain a factor $z^2 - 1$, and therefore does not vanish at $z = 1$, is $(2z)^l l!(z^2 - 1)^0$. Putting $z = 1$ in Eq. (41) therefore shows that $c_l = 2^l l!$, thus completing the proof of Eq. (40).

Example

Use Rodrigues' formula to show that

$$I_l = \int_{-1}^{1} P_l(z) P_l(z)\, dz = \frac{2}{2l+1}. \qquad (42)$$

Answer

This is obvious for $l = 0$, and so we assume $l \geq 1$. Then, by Rodrigues' formula

$$I_l = \frac{1}{2^{2l}(l!)^2} \int_{-1}^{1} \left[\frac{d^l (z^2-1)^l}{dz^l} \right] \left[\frac{d^l (z^2-1)^l}{dz^l} \right] dz.$$

Repeated integration by parts, with all boundary terms vanishing, reduces this to

$$\begin{aligned} I_l &= \frac{(-1)^l}{2^{2l}(l!)^2} \int_{-1}^{1} (z^2-1)^l \frac{d^{2l}}{dz^{2l}} (z^2-1)^l \, dz \\ &= \frac{(2l)!}{2^{2l}(l!)^2} \int_{-1}^{1} (1-z^2)^l \, dz. \end{aligned}$$

If we write

$$K_l = \int_{-1}^{1} (1 - z^2)^l \, dz,$$

then integration by parts (taking a factor 1 as the second part) gives

$$K_l = \int_{-1}^{1} 2lz^2 (1 - z^2)^{l-1} dz.$$

Writing $2lz^2$ as $2l - 2l(1 - z^2)$ we obtain

$$\begin{aligned} K_l &= 2l \int_{-1}^{1} (1 - z^2)^{l-1} dz - 2l \int_{-1}^{1} (1 - z^2)^l \, dz, \\ &= 2lK_{l-1} - 2lK_l \end{aligned}$$

and hence the recurrence relation
$(2l + 1)K_l = 2lK_{l-1}$. We therefore find

$$\begin{aligned} K_l &= \frac{2l}{2l+1} \frac{2l-2}{2l-1} \cdots \frac{2}{3} K_0 \\ &= 2^l l! \frac{2^l l!}{(2l+1)!} 2 = \frac{2^{2l+1} (l!)^2}{(2l+1)!}. \end{aligned}$$

Another property of the $P_l(z)$ is their mutual orthogonality

$$\int_{-1}^{1} P_l(z) P_k(z)\, dz = 0 \quad \text{if } l \neq k. \tag{43}$$

Generating function for Legendre polynomials

The generating function for, say, a series of functions $f_n(z)$ for $n = 0, 1, 2, \ldots$ is a function $G(z, h)$, containing as well as z a dummy variable h, such that

$$G(z, h) = \sum_{n=0}^{\infty} f_n(z) h^n,$$

i.e. $f_n(z)$ is the coefficient of h^n in the expansion of G in powers of h.

Let us consider the functions $P_n(z)$ defined by the equation

$$G(z, h) = (1 - 2zh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z) h^n. \quad (44)$$

The functions so defined are identical to the Legendre polynomials and the function $(1 - 2zh + h^2)^{-1/2}$ is the generating function for them.

In the following, $dP_n(z)/dz$ will be denoted by P_n'. First, we differentiate the defining equation Eq. (44) with respect to z to get

$$h(1-2zh+h^2)^{-3/2} = \sum P_n' h^n. \tag{45}$$

Also. we differentiate Eq. (44) with respect to h to yield

$$(z-h)(1-2zh+h^2)^{-3/2} = \sum nP_n h^{n-1}. \tag{46}$$

Eq. (45) can be written suing Eq. (44) as

$$h\sum P_n h^n = (1-2zh+h^2)\sum P_n' h^n,$$

and thus equating coefficients of h^{n+1} we obtain the recurrence relation

$$P_n = P_{n+1}' - 2zP_n' + P_{n-1}'. \tag{47}$$

Eqs. (45) and (46) can be combined as

$$(z-h)\sum P_n' h^n = h\sum nP_n h^{n-1},$$

from which the coefficient of h^n yields a second recurrence relation

$$zP_n' - P_{n-1}' = nP_n. \tag{48}$$

Eliminating P'_{n-1} between Eqs. (47) and (48) gives the further result

$$(n+1)P_n = P'_{n+1} - zP'_n. \tag{49}$$

If we now take the result Eq. (49) with n replaced by $n-1$ and add z times Eq. (48) to it, we obtain

$$(1-z^2)P'_n = n(P_{n-1} - zP_n).$$

Finally, differentiating both sides with respect to z and using Eq. (48) again, we find

$$\begin{aligned}(1-z^2)P''_n - 2zP'_n &= n[(P'_{n-1} - zP'_n) - P_n] \\ &= n(-nP_n - P_n) \\ &= -n(n+1)P_n,\end{aligned}$$

and so the P_n defined by Eq. (44) satisfies Legendre's equation. It only remains to verify the normalization. This is easily done at $z=1$ when G becomes

$$G(1,h) = [(1-h)^2]^{-1/2} = 1 + h + h^2 + \cdots,$$

and thus all the P_n so defined have $P_n(1) = 1$.

Bessel's equation

Bessel's equation has the form

$$z^2 y'' + z y' + (z^2 - \nu^2) y = 0, \tag{50}$$

where the parameter ν is a given number, which we may take as ≥ 0 with no loss of generality. In Bessel's equation, z is usually a multiple of a radial distance and therefore ranges from 0 to ∞.

Rewriting Eq. (50) we have

$$y'' + \frac{1}{z} y' + \left(1 - \frac{\nu^2}{z^2}\right) y = 0. \tag{51}$$

By inspection $z = 0$ is a regular singular point; hence we try a solution of the form $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$. Substituting this into Eq. (51) and multiplying the resulting equation by $z^{2-\sigma}$, we obtain

$$\sum_{n=0}^{\infty} \left[(\sigma + n)(\sigma + n - 1) + (\sigma + n) - \nu^2\right] a_n z^n$$

$$+ \sum_{n=0}^{\infty} a_n z^{n+2} = 0,$$

which simplifies to

$$\sum_{n=0}^{\infty}[(\sigma+n)^2-\nu^2]a_n z^n+\sum_{n=0}^{\infty}a_n z^{n+2}=0.$$

Considering coefficients of z^0 we obtain the indicial equation

$$\sigma^2-\nu^2=0,$$

so that $\sigma=\pm\nu$. For coefficients of higher powers of z we find

$$\begin{aligned}\left[(\sigma+1)^2-\nu^2\right]a_1 &= 0, &(52)\\ \left[(\sigma+n)^2-\nu^2\right]a_n+a_{n-2} &= 0 \quad \text{for } n\geq 2. &(53)\end{aligned}$$

Substituting $\sigma=\pm\nu$ into Eqs. (52) and (53) we obtain the recurrence relations

$$\begin{aligned}(1\pm 2\nu)a_1 &= 0, &(54)\\ n(n\pm 2\nu)a_n+a_{n-2} &= 0 \quad \text{for } n\geq 2. &(55)\end{aligned}$$

General solution for non-integer ν

If ν is a non-integer, then in general the two roots of the indicial equation, $\sigma_1 = \nu$ and $\sigma_2 = -\nu$, will not differ by an integer, and we may obtain two linearly independent solutions. Special cases od arise, however, when $\nu = m/2$ for $m = 1, 3, 5, \ldots$, and $\sigma_1 - \sigma_2 = 2\nu = m$ is an (odd positive) integer. For such cases, we may always obtain a solution in the form of Frobenius series corresponding to the larger root $\sigma_1 = \nu = m/2$. For the smaller root $\sigma_2 = -\nu = -m/2$, however, we must determine whether a second Frobenius series solution is possible by examining the recurrence relation Eq. (55), which reads

$$n(n-m)a_n + a_{n-2} = 0 \quad \text{for } n \geq 2.$$

Since m is an odd positive integer in this case, we can use this recurrence relation (starting with $a_0 \neq 0$) to calculate $a_2, a_4, a_6, \ldots$. It is therefore possible to find a second solution in the form of a Frobenius series corresponding to the smaller root σ_2.

Therefore, in general, for non-integer ν we have from Eqs. (54) and (55)

$$\begin{aligned} a_n &= -\frac{1}{n(n \pm 2\nu)} a_{n-2} \text{ for } n = 2, 4, 6, \dots, \\ &= 0 \quad \text{for } n = 1, 3, 5, \dots . \end{aligned}$$

Setting $a_0 = 1$ in each case, we obtain the two solutions

$$\begin{aligned} y_{\pm\nu}(z) &= z^{\pm\nu} \left[1 - \frac{z^2}{2(2 \pm 2\nu)} + \right. \\ & \left. \frac{z^4}{2 \times 4(2 \pm 2\nu)(4 \pm 2\nu)} - \dots \right] . \end{aligned}$$

We set

$$a_0 = \frac{1}{2^{\pm\nu}\Gamma(1 \pm \nu)},$$

where $\Gamma(x)$ is the gamma function.

The two solutions of Eq. (50) are then written as $J_\nu(z)$ and $J_{-\nu}(z)$, where

$$\begin{aligned} J_\nu(z) &= \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \left[1 - \frac{1}{\nu+1}\left(\frac{z}{2}\right)^2\right. \\ &\qquad \left. + \frac{1}{(\nu+1)(\nu+2)}\frac{1}{2!}\left(\frac{z}{2}\right)^4 - \cdots\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}; \quad (56) \end{aligned}$$

and replacing ν by $-\nu$ gives $J_{-\nu}(z)$. The functions $J_\nu(z)$ and $J_{-\nu}(z)$ are called Bessel functions of the first kind, of order ν. Since the first term of each series is a finite non-zero multiple of z^ν and $z^{-\nu}$ respectively, if ν is not an integer, $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent. Therefore, for non-integer ν the general solution of Bessel's equation Eq. (50) is

$$y(z) = c_1 J_\nu(z) + c_2 J_{-\nu}(z). \qquad (57)$$

General solution for integer ν

Let us consider the case when $\nu = 0$, so that the two solutions to the indicial equation are equal, and we obtain one solution in the form of a Frobenius series. From Eq. (56), this is given by

$$\begin{aligned} J_0(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{2^{2n} n! \Gamma(1+n)} \\ &= 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} + \cdots . \end{aligned}$$

In general, however, if ν is a positive integer then the solutions to the indicial equation differ by an integer. For the larger root, $\sigma_1 = \nu$, we may find a solution $J_\nu(z)$ for $\nu = 1, 2, 3, \ldots$, in the form of a Frobenius series given by Eq. (56).

For the smaller root $\sigma_2 = -\nu$, however, the recurrence relation Eq. (55) becomes

$$n(n-m)a_n + a_{n-2} = 0 \text{ for } n \geq 2,$$

where $m = 2\nu$ is now an even positive integer, i.e. $m = 2, 4, 6, \ldots$.

Starting with $a_0 \neq 0$, we may calculate $a_2, a_4, a_6, \ldots$, but we see that when $n = m$ the coefficient a_n is infinite, and the method fails to produce a second solution.

By replacing ν by $-\nu$ in the definition of $J_\nu(z)$, Eq. (56), we have, for integer ν,

$$J_{-\nu}(z) = (-1)^\nu J_\nu(z)$$

and hence that $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly dependent. One then defines the function

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \qquad (58)$$

which is called a Bessel's function of the second kind or order ν. For non-integer ν, $Y_\nu(z)$ is linearly independent of $J_\nu(z)$. It may also be shown that the Wronskian of $J_\nu(z)$ and $Y_\nu(z)$ is non-zero for all values of ν. Hence $J_\nu(z)$ and $Y_\nu(z)$ always constitute a pair of independent solutions. The expression Eq. (58) does, however, become an indeterminate form $0/0$ when ν is an integer.

This is so because for integer ν we have $\cos\nu\pi = (-1)^\nu$ and $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$. For integer ν, we set

$$Y_\nu(z) = \lim_{\mu\to\nu}\left[\frac{J_\mu(z)\cos\mu\pi - J_{-\mu}(z)}{\sin\mu\pi}\right], \qquad (59)$$

which gives a linearly independent second solution for integer ν. Therefore, we may write the general solution of Bessel's equation, valid for all ν, as

$$y(z) = c_1 J_\nu(z) + c_2 Y_\nu(z). \qquad (60)$$

Properties of Bessel functions

Example

Prove the recurrence relation

$$\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z). \qquad (61)$$

Answer

From the power series definition Eq. (56) of $J_\nu(z)$, we obtain

$$\begin{aligned}
\frac{d}{dz}[z^\nu J_\nu(z)] &= \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2\nu+2n}}{2^{\nu+2n}n!\Gamma(\nu+n+1)} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2\nu+2n-1}}{2^{\nu+2n-1}n!\Gamma(\nu+n)} \\
&= z^\nu \sum_{n=0}^{\infty}\frac{(-1)^n z^{(\nu-1)+2n}}{2^{(\nu-1)+2n}n!\Gamma(\nu-1+n+1)} \\
&= z^\nu J_{\nu-1}(z).
\end{aligned}$$

Similarly, we have

$$\frac{d}{dz}[z^{-\nu}J_\nu(z)] = -z^{-\nu}J_{\nu+1}(z). \tag{62}$$

From Eqs. (61) and (62) the remaining recurrence relations may be derived. Expanding out the derivative on the LHS of Eq. (61) and dividing through by $z^{\nu-1}$, we obtain the relation

$$zJ'_\nu(z) + \nu J_\nu(z) = zJ_{\nu-1}(z). \tag{63}$$

Similarly, by expanding out the derivative on the LHS of Eq. (62), and multiplying through by $z^{\nu+1}$, we find

$$zJ'_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z). \tag{64}$$

Adding Eq. (63) and Eq. (64) and dividing through by z gives

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z). \tag{65}$$

Finally, subtracting Eq. (64) from Eq. (63) and dividing through by z gives

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z). \tag{66}$$

Mutually orthogonality of Bessel functions

By definition, the function $J_\nu(z)$ satisfies Bessel's equation Eq. (50),

$$z^2 y'' + z y' + (z^2 - \nu^2) y = 0.$$

Let us instead consider the functions $f(z) = J_\nu(\lambda z)$ and $g(z) = J_\nu(\mu z)$, which respectively satisfy the equations

$$\begin{aligned} z^2 f'' + z f' + (\lambda^2 z^2 - \nu^2) f &= 0, \qquad (67) \\ z^2 g'' + z g' + (\mu^2 z^2 - \nu^2) g &= 0. \qquad (68) \end{aligned}$$

Example

Show that $f(z) = J_\nu(\lambda z)$ satisfies Eq. (67).

Answer

If $f(z) = J_\nu(\lambda z)$ and we write $w = \lambda z$, then

$$\frac{df}{dz} = \lambda \frac{dJ_\nu(w)}{dw} \quad \text{and} \quad \frac{d^2 f}{dz^2} = \lambda^2 \frac{d^2 J_\nu(w)}{dw^2}.$$

When these expressions are substituted, the LHS of Eq. (67) becomes

$$z^2\lambda^2 \frac{d^2 J_\nu(w)}{dw^2} + z\lambda \frac{dJ_\nu(w)}{dw} + (\lambda^2 z^2 - \nu^2)J_\nu(w)$$
$$= w^2 \frac{d^2 J_\nu(w)}{dw^2} + w \frac{dJ_\nu(w)}{dw} + (w^2 - \nu^2)J_\nu(w).$$

But, from Bessel's equation itself, this final expression is equal to zero, thus verifying that $f(z)$ does satisfy Eq. (67).

Now multiplying Eq. (68) by $f(z)$ and Eq. (67) by $g(z)$ and subtracting gives

$$\frac{d}{dz}[z(fg' - gf')] = (\lambda^2 - \mu^2)zfg, \qquad (69)$$

where we have used the fact that

$$\frac{d}{dz}[z(fg' - gf')] = z(fg'' - gf'') + (fg' - gf').$$

By integrating Eq. (69) over any given range $z = a$ to $z = b$, we obtain

$$\begin{aligned}\int_a^b zf(z)g(z)\,dz \quad &= \quad \frac{1}{\lambda^2 - \mu^2}\,[zf(z)g'(z) \\ &\qquad -zg(z)f'(z)]_a^b\,,\end{aligned}$$

which, on setting $f(z) = J_\nu(\lambda z)$ and $g(z) = J_\nu(\mu z)$, becomes

$$\begin{aligned}\int_a^b zJ_\nu(\lambda z)J_\nu(\mu z)\,dz \quad &= \quad \frac{1}{\lambda^2 - \mu^2}\,[\mu z J_\nu(\lambda z)J'_\nu(\mu z) \\ &\qquad -\lambda z J_\nu(\mu z)J'_\nu(\lambda z)]_a^b\,. \quad (70)\end{aligned}$$

If $\lambda \neq \mu$ and the interval $[a, b]$ is such that the expression on the RHS of Eq. (70) equals zero we obtain the orthogonality condition

$$\int_a^b z J_\nu(\lambda z) J_\nu(\mu z)\, dz = 0. \tag{71}$$

Generating function for Bessel functions

The Bessel functions $J_\nu(z)$, where ν is an integer, cab be described by a generating function, which is given by

$$G(z,h) = \exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) h^n. \tag{72}$$

By expanding the exponential as a power series, it can be verified that the functions $J_n(z)$ defined by Eq. (72) are Bessel functions of the first kind.

Example

Use the generating function Eq. (72) to prove, for integer ν, the recurrence relation Eq. (66), i.e.

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z).$$

Answer

Differentiating $G(z, h)$ with respect to h, we obtain

$$\frac{\partial G(z,h)}{\partial h} = \frac{z}{2}\left(1 + \frac{1}{h^2}\right) G(z,h) = \sum_{-\infty}^{\infty} n J_n(z) h^{n-1},$$

which can be written using Eq. (72) again as

$$\frac{z}{2}\left(1 + \frac{1}{h^2}\right) \sum_{-\infty}^{\infty} J_n(z) h^n = \sum_{-\infty}^{\infty} n J_n(z) h^{n-1}.$$

Equating coefficients of h^n we obtain

$$\frac{z}{2}\left[J_n(z) + J_{n+2}(z)\right] = (n+1) J_{n+1}(z),$$

which on replacing n by $\nu - 1$ gives the required recurrence relation.