

Remarks on some basic issues in quantum mechanics*

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Two-way interferometers with which-way detectors are not only of importance in physical research, they are also a useful teaching device. A number of basic issues can be illustrated and discussed, even at the level of undergraduate teaching. Among these issues are: the physical meaning of a state vector; entangled systems; Einstein-Podolsky-Rosen correlations; statistical operators and the as-if realities associated with them; quantum erasure; Schrödinger’s cat; and, finally, wave-particle duality.

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1. Introduction

1.1. Motivation, objective, and outline

More than once, I have come across statements such as “The formalism of quantum mechanics is well established, and there are a number of good textbooks that treat the subject satisfactorily. But the physical interpretation of the formalism is still under debate, and there are a number of questions that still lack a satisfactory answer.” I disagree for two reasons.

First, I cannot imagine teaching the formalism independently of the interpretation. Before one can write the Schrödinger equation on the blackboard, one must have already explained the significance of the wave function that is differentiated.¹ Otherwise the equation is meaningless and has no right to appear in a physics lecture.

Second, all the allegedly unanswerable questions show up in discussions of measurements on quantum objects, and I am convinced that there are no questions in this context that cannot be dealt with in an introductory course on quantum mechanics. It is the objective of these notes to supply a concise exposition that is based on which-way detection in simple two-way interferometers.

After a brief repetition of the basics of two-way interferometers in section 2, we turn to which-way detection in section 3. There we meet Einstein-Podolsky-Rosen correlations and recall, in passing, what is meant by phrases like “the system is in a certain state.” In section 4, three experimental set-ups are sketched that illustrate different methods of incorporating devices for which-way detection into standard interferometers. After dealing with the notion of entanglement in section 5, we are ready for a discussion of the physical meaning of the statistical operator (section 6). The ground is then prepared for a short lesson about quantum erasure in section 7. Some

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¹Of course, the interpretation of the wave function, say, may be subject to refinements when our understanding evolves, but there is an agreed-upon interpretation at each stage of the development.

objections are shown to be invalid, and that brings up the subject of state reduction (section 8). When offering additional remarks on quantum erasure, in section 9, we run into Schrödinger’s cat. Finally, section 10 deals with wave-particle duality in a quantitative manner.

The material is well suited for an undergraduate course because the mathematical aspects are rather elementary. Some basic knowledge about Hilbert space vectors, self-adjoint and unitary operators, and the like suffices for sections 2–9. In section 10, some properties of the trace-class norm are made use of; although this might not be standard course material, it can be supplied easily.

The presentation is entirely in the language of quantum kinematics, no dynamical equations are employed. The students must only know that the transition from “before” to “after” is effected by a unitary transformation. Detailed temporal evolution plays no role, but the causal order in which things happen is relevant, of course.

This is not to say that time dependences can always be ignored in discussions of quantum measurements. All measurements take time (and they also happen inside a certain spatial region). Some phenomena, such as the quantum Zeno effect, can only be understood by paying careful attention to the evolution. This subject matter is, however, beyond the scope of these notes, and I refer the reader to A. Schenzle’s instructive article [1].

Occasionally, I remark on things that will strike the experienced reader as rather elementary. These remarks are meant for those undergraduates who study the material without a teacher’s guidance.

1.2. A confession

In the opening paragraphs I have already offered a personal opinion. Perhaps I should confess more thoroughly where I stand.

Quantum mechanics, as I understand it, is solidly founded on experimental findings² and the theoretical conclusions drawn by M. Planck, A. Einstein, N. Bohr, W. Heisenberg, W. Pauli, E. Schrödinger, and P. A. M. Dirac, to name the main contributors. One can learn the subject from the classic textbooks by P. A. M. Dirac [3], D. Bohm [4], and K. Gottfried [5]; J. von Neumann’s book [6] puts particular emphasis on mathematical aspects; a modern text that is much to my liking is the one by L. E. Ballentine [7].³ In addition, E. Schrödinger’s seminal essay of 1935 [8] — best remembered for a

²Attempts to infer quantum mechanics from some mathematical statements are misguided, in my opinion, irrespective of how convincing the arguments may appear at first glance. A recent example is F. H. Fröhner’s undertaking [2] who believes, so it seems, that all of quantum mechanics follows from the Riesz-Fejér theorem.

³There are, of course, other books worthy of recommendation. Nothing is implied by not mentioning them.

marginal issue, the cat example — is recommended reading, and so are G. Süßmann’s [9] and N. G. van Kampen’s [10] remarks on quantum measurements.

Quantum mechanics works (and so does its relativistic extension, renormalized quantum field theory). During the seven decades since its conception, we have not become aware of a single observational fact in disagreement with quantum mechanical predictions. Although this large body of evidence lends strong support to the judgment that quantum mechanics provides for a consistent picture of the physical world, the logical possibility of a future failure is not excluded, of course. As soon as such a failure will have occurred, we’ll be living through exciting times and something profoundly new will be learned. For the time being, however, there is no need for modifications of quantum mechanics.

Some are bound to disagree with the last sentence because they feel uneasy with the fundamentally probabilistic world view of quantum mechanics, and an intrinsically deterministic universe is philosophically more appealing to them. The pseudo-classical mechanics invented by D. Bohm [11,12] is the prime example. By construction, Bohmian mechanics agrees with quantum mechanics as far as its experimentally testable aspects are concerned, and has additional elements (namely hypothetical particle trajectories) that are of no consequence. In addition, these trajectories possess extremely implausible properties [13–15] that invalidate the realistic interpretation intended by D. Bohm.

Quantum mechanics deals with the behavior of atomic systems and, in particular, with measurements on them. For the concept of a measurement to be meaningful, it is necessary — as W. Heisenberg and N. Bohr have emphasized — that the physical world can be divided into the atomic system under study, whose quantum properties are important, and the rest, whose quantum properties are irrelevant. The measuring apparatus (a photographic plate, say, or a Geiger counter) is part of the “rest.” Of course, this does not deny that the apparatus exhibits quantum features itself — the contrary is true: the photographic process and the mechanism of the Geiger counter rely on quantum processes — but only that these aspects are presently irrelevant.

Consider, for example, a magnetic silver atom that passes through the inhomogeneous magnetic field of a Stern-Gerlach apparatus and then hits a glass plate; photographic development eventually reveals whether the atom was deflected up or down. Here we are studying the quantum properties of atomic magnetism. Do the quantum mechanical details of the chemistry of the photographic process matter? No, they don’t. Our conclusions concerning the magnetic atoms are not altered if we use another detection process instead of the glass plate.

We can, therefore, ignore the technical details of the detection and simply speak of a detector for the atoms. In a next step we recognize that the detector actually determines the position of the atom. So we learn that the crucial element of the apparatus is the inhomogeneous

magnetic field because it encodes the information about the magnetic moment in the center-of-mass motion. This leads us to regarding one center-of-mass coordinate as the basic physical quantity carrying the measurement result. That coordinate is then the quantum *marker*⁴ in this example. The glass plate et cetera amplifies the datum recorded by the marker, and as a result of the amplification we can register the outcome and tell our colleagues what has been found.

One could insist that the measurement is completed only after the amplification has turned the information stored in the center-of-mass state (the *marker* state) into a macroscopically recognizable signal. But it is clear that the essential step is the one in which the information is encoded in the center-of-mass motion.

The amplification is part of the final measurement of the marker state, the so-called “reading of the marker.” Prior to the amplification we could, in this example, manipulate the center-of-mass motion with the intention to read the marker in a different way. This freedom is particularly important in sections 7 and 9.

The possibility of dividing the physical world into the quantum system of interest and the (quasi-)classical rest is a fundamental empirical fact. Without this division, one could not speak sensibly about measurements on quantum systems, because there wouldn’t be any classical systems for reference. The division is not only possible, it is also necessary.⁵

Whereas it is a healthy attitude of working physicists to simply accept this division as an empirical fact, one can, and should, ask

Can quantum mechanics explain why the
quantum/classical division is possible? (1)

For reasons such as the ones discussed in section 9.3, I am confident that the answer will be “yes” eventually. But, admittedly, the case is not closed as yet. There are others who have seemingly convinced themselves that quantum mechanics does not rise to this challenge. The late J. Bell was arguably the most outspoken advocate of this point of view. He put “measurement” on his list of forbidden words [16,17] — in fact, he declared it “the very worst word on this list” — and claimed that quantum mechanics “carries within itself the seed of its own destruction” [18]. Myself, I find it impossible to agree with J. Bell and recognize my own convictions in the replies by N. G. van Kampen [19], by the late R. Peierls [20], and by K. Gottfried [21].

⁴What I call the *marker* here is alternatively termed *meter* or even *detector* in the literature. Since macroscopic devices are frequently associated both with meters (such as thermometers) and with detectors (such as Geiger counters), I suggest to use the less provocative word “marker” instead.

⁵As a consequence, I have no use for a “wave function of the universe.” But that’s a side issue.

The interference of alternatives is characteristic of quantum systems; classical alternatives do not interfere.⁶ It should be clear that there is a border regime where quantum interferences are faint, and therefore a sharp boundary cannot exist. Nevertheless, one is almost always sure whether certain degrees of freedom are on the quantum side or on the classical side. And in the case of doubt, one plays it safe and enlarges the quantum system — one “shifts the *Heisenberg cut*” as the phrase goes.

Quantum mechanics has been accused of being incomplete or inexact or simply ill-defined because there is no formal procedure that would draw a unique dividing line between the quantum system and the classical rest. These charges are unfounded. It is the nature of the Heisenberg cut that it can be shifted to some extent, but not arbitrarily. There are always certain degrees of freedom that are undoubtedly on one side of the cut or on the other. One can reasonably, and in a generally agreed-upon way, speak of day and night without ever assigning to dawn a uniquely defined instant in time.⁷

An affirmative answer to (1) needs a mechanism that explains why alternatives have different interference properties on the opposite sides of the quantum/classical border. In one approach, exemplified by the Ghirardi-Rimini-Weber scheme [22], the dynamical equations of quantum mechanics are modified; in view of their ad-hoc nature, however, such schemes are hardly convincing. I have much more sympathy for the other, conservative, approach that does not give up lightly what has been hard won; it takes quantum mechanics at its face value and searches for the said mechanism by investigating the quantum properties of systems with very many degrees of freedom. The *decoherence* process studied by W. H. Zurek [23] and others is very likely an essential ingredient.

An important aspect of the question (1) concerns the occurrence of factual *events*, that is: local interactions that have surely happened. Rain is falling independently of any manipulation by an experimenter. The formation of each individual rain drop depends on quantum processes at its initial stage; once it is formed, a drop is a classical object and undoubtedly in existence. More elementary than this formation process are scattering events in which a few particles participate only. In a recent proposal by R. Haag [24], the emphasis is shifted from measurements and their results to events as (one of) the fundamental concept(s). This intriguing program has not been worked out as yet to the extent necessary for a final judgment; I do think, however, that it opens a new front at which (1) can be attacked.

⁶In anticipation of the discussion in section 9.3.2 I note that, therefore, there are no physical observables associated with selfadjoint operators that are sensitive to interferences between classical alternatives.

⁷I owe this telling analogy to H.-J. Briegel.

2. Two-way interferometers

In a double-slit interferometer the pattern on the screen originates in the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|S_1\rangle + |S_2\rangle), \quad (2)$$

where $|S_1\rangle$, $|S_2\rangle$ symbolize the amplitudes of slit 1 and slit 2, respectively. These amplitudes are orthonormal,

$$\langle S_j | S_k \rangle = \delta_{jk}, \quad (3)$$

and thus $|\psi\rangle$ is properly normalized. The corresponding statistical operator (*vulgo* the density matrix)

$$\begin{aligned} \rho_Q^{(0)} = |\psi\rangle\langle\psi| &= \frac{1}{2} \left(\underbrace{|S_1\rangle\langle S_1|}_{\text{slit 1}} + \underbrace{|S_2\rangle\langle S_2|}_{\text{slit 2}} \right) \\ &+ \frac{1}{2} \left(\underbrace{|S_1\rangle\langle S_2| + |S_2\rangle\langle S_1|}_{\text{cross terms}} \right) \end{aligned} \quad (4)$$

is half the sum of the single-slit contributions plus cross terms, which give rise to the double-slit interference pattern.

It is clear that this mathematical structure is common to all (symmetric) two-way interferometers, such as double-slit interferometers (for light, electrons, neutrons, or atoms), or Mach-Zehnder interferometers, both of the optical and of the neutron kind, or biprism interferometers, with light or electrons, or Stern-Gerlach interferometers for magnetic atoms, or Ramsey-Bordé interferometers for two-level atoms, or photon-pair interferometers, et cetera.⁸ We shall continue to speak of slit 1, slit 2, the screen, . . . , but the reader should keep in mind that the double-slit interferometer is just a stand-in for all two-way interferometers. Further, rather than speaking of particles or waves we use the noun *quanton* (suggested by M. Bunge as reported by J.-M. Lévy-Leblond [26]) as a generic term for the interfering quantum object (photon, electron, neutron, atom, . . .). The subscript Q in (4) anticipated this terminology.

In the spirit of this implicit generality we shall not pay attention to the subtleties of the pattern formation on the screen (such as the keeping apart of the single-slit diffraction pattern from the double-slit interference pattern that we are interested in, or the necessity to integrate over the various arrival times). Instead we exhibit the interference pattern in the probability $p(\phi)$ of finding the superposition

$$|S(\phi)\rangle = \frac{1}{\sqrt{2}}(|S_1\rangle + |S_2\rangle e^{i\phi}), \quad (5)$$

⁸By now there is an extensive literature on interferometers. The review articles in reference [25] report the state of the art.

where ϕ is the interferometric phase difference.⁹ In the state characterized by (2) or (4), this probability is

$$\begin{aligned} p^{(0)}(\phi) &= |\langle S(\phi) | \psi \rangle|^2 = \text{tr}_Q \left\{ |S(\phi)\rangle\langle S(\phi)| \rho_Q^{(0)} \right\} \\ &= \frac{1}{2}(1 + \cos\phi). \end{aligned} \quad (6)$$

Its ϕ dependence exhibits interference fringes, and since the extreme values of $p^{(0)}(\phi)$ are

$$p_{\max}^{(0)} = 1, \quad p_{\min}^{(0)} = 0, \quad (7)$$

the visibility $\mathcal{V}^{(0)}$ of these fringes,

$$\mathcal{V}^{(0)} = \frac{p_{\max}^{(0)} - p_{\min}^{(0)}}{p_{\max}^{(0)} + p_{\min}^{(0)}} = 1, \quad (8)$$

is as large as it can possibly be.

Perhaps unnecessarily I note that the piece of information “the quanton has a 50:50 chance of passing through slit 1 or slit 2” is not sufficient to determine $\rho_Q^{(0)}$ of (4). All statistical operators of the form

$$\begin{aligned} \rho_{Q,\epsilon}^{(0)} &= \frac{1}{2}(|S_1\rangle\langle S_1| + |S_2\rangle\langle S_2|) \\ &+ \frac{1}{2}(|S_1\rangle\epsilon^*\langle S_2| + |S_2\rangle\epsilon\langle S_1|) \end{aligned} \quad (9)$$

are consistent with this information, not only the $\epsilon = 1$ version of (4). The positivity of $\rho_{Q,\epsilon}^{(0)}$ requires $|\epsilon| \leq 1$ but no further restrictions apply to the complex number ϵ . Knowledge of the degree of coherence between the two quantum alternatives “through slit 1” and “through slit 2” as well as of their phase relation is needed to fix the value of ϵ .

3. Which-way detection, EPR correlations

Now let us consider the more complicated situation in which the interferometer is supplemented by a device for which-way (WW) detection. A few quantum degrees of freedom of this device constitute the WW marker.¹⁰ Eventually an amplification will enable the experimenter to read the marker, thereby extracting the WW information stored in the marker state.

The quanton interacts with the marker. Ideally, two orthonormal states $|M_1\rangle$ and $|M_2\rangle$ of the marker get then correlated with the amplitudes $|S_1\rangle$ and $|S_2\rangle$ of the two

⁹The value of ϕ is determined, for example, by the site where the quanton hits the screen of a double-slit interferometer, or by the difference in the optical path lengths of a Mach-Zehnder interferometer; see section 4.

¹⁰More than “a few” lead to the situation discussed in section 9.3.

slits, so that the combined system of interfering quanton and WW marker has the state vector

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|M_1, S_1\rangle + |M_2, S_2\rangle) \quad (10)$$

and a corresponding statistical operator¹¹

$$\begin{aligned} P &= |\Psi\rangle\langle\Psi| \\ &= \frac{1}{2}(|M_1, S_1\rangle\langle M_1, S_1| + |M_2, S_2\rangle\langle M_2, S_2|) \\ &\quad + \frac{1}{2}(|M_1, S_1\rangle\langle M_2, S_2| + |M_2, S_2\rangle\langle M_1, S_1|). \end{aligned} \quad (11)$$

Again we recognize contributions that refer to one way only or to both ways.

The purpose of the WW marker should be obvious. If we find the marker in state $|M_1\rangle$ then the quanton is surely in state $|S_1\rangle$, and likewise for $|M_2\rangle$ and $|S_2\rangle$. In other words:

$$\begin{aligned} &\text{Knowing that the WW marker is in state} \\ &|M_1\rangle \text{ or } |M_2\rangle \text{ is tantamount to knowing} \\ &\text{that the quanton is in state } |S_1\rangle \text{ or } |S_2\rangle, \\ &\text{respectively.} \end{aligned} \quad (12)$$

Therefore, it is now possible to manipulate the quanton — for example, by making a measurement on it — without losing the information whether the quanton is in state $|S_1\rangle$ or $|S_2\rangle$, because this WW information is stored in the WW marker.

Correlations of the Einstein-Podolsky-Rosen (EPR) type [27] are exploited here. Their reality can be demonstrated experimentally, possibly in the manner Alice and Bob do it in section 8. But one must not be led astray by extreme operationalism: The EPR correlations exist irrespective of their observation by a human observer, they are a physical property of the combined quanton-marker system in the state $|\Psi\rangle$ of (10).¹²

The phrase “the quanton is in state $|S_1\rangle$ ” is shorthand for “the statistical properties of (future) measurements on the quanton are correctly predicted by the statistical operator $\rho_Q = |S_1\rangle\langle S_1|$.” — This minimalistic interpretation of state vectors and statistical operators is quite sufficient. One might, of course, try to go beyond this minimalism and give additional ontological meaning to the state vector, perhaps responding to a philosophical impetus. In doing so, one should however remember N. G. van Kampen’s caveat: Whoever endows $|S_1\rangle$ (or any other state vector [...]) with more meaning than is needed for computing observable

phenomena is responsible for the consequences (Theorem IV in reference [10]). — Similarly, the phrase “the WW marker is found in $|M_1\rangle$ ” abbreviates a statement such as “the result of a measurement of the observable $M = |M_1\rangle\langle M_1| - |M_2\rangle\langle M_2|$ is the eigenvalue $M' = 1$.”¹³

Having the WW information safely stored we can now think of looking for the interference pattern. Inasmuch as knowing the way gives evidence of the quanton’s particle aspects, whereas the observation of interference fringes manifests the quanton’s wave aspects, we expect that complementarity prevents us from getting both. Indeed, the quanton’s statistical operator that is needed in the trace of (6), obtained by tracing P over the marker’s degree(s) of freedom,

$$\rho_Q = \text{tr}_M \{P\} = \frac{1}{2}(|S_1\rangle\langle S_1| + |S_2\rangle\langle S_2|), \quad (13)$$

no longer contains the cross terms of (4); now the pattern

$$p(\phi) = \text{tr}_Q \{ |S(\phi)\rangle\langle S(\phi)| \rho_Q \} = \frac{1}{2} \quad (14)$$

is just the fringeless background of the single-slit contributions.

The fringes disappear whether we read the WW marker to learn about the way or not. The transition from $\rho_Q^{(0)}$ of (4) to ρ_Q of (13) (termed the “collapse of the wave function” in some jargon) is not caused by the acquisition of human knowledge; surely, “it has nothing to do with the mind” (J. A. Wheeler as quoted by R. Haag [33]).

4. Examples of interferometers with which-way marking

A first example for an interferometer with a which-way detection device, the thought experiment of reference [34], is sketched in figure 1. The two slits of a Young interferometer are illuminated by atomic de Broglie waves. Prior to reaching the slits, the atoms pass through resonators; the de Broglie waves are precollimated (by some equipment not shown in the figure) so that the partial amplitudes fit through the entrance and exit holes of the

¹³“Measurement” means a traditional *von Neumann measurement*, where the possible outcomes are the eigenvalues of the observable in question. Y. Aharonov and L. Vaidman, in collaboration with D. Z. Albert and J. Anandan, have recently introduced the intriguing and useful concepts of *weak measurement* [29,30] and *protective measurement* [31,32], where other properties of the measured observable are more relevant. At their final stage, however, a von Neumann measurement determines the position, say, of a pointer variable. — I am grateful for the enjoyable and instructive discussions with Y. Aharonov on these matters.

¹¹The letter P is the upper case of p , not of p .

¹²N. D. Mermin has recently advertised his “Ithaca interpretation” [28] whose central theme is the assertion that correlations (of the EPR type) are real and that they are the only real thing. I have much sympathy for this point of view.

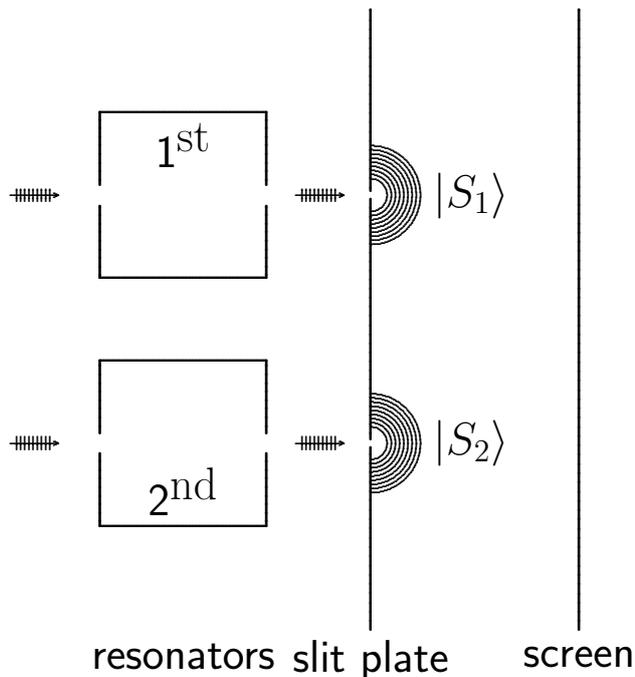


FIG. 1. Atomic de Broglie waves, indicated by symbolic wave trains, are collimated so that they pass through resonators before reaching the slits of a Young’s double-slit interferometer. At most one atom at a time is in the apparatus. The atom emits a photon into the resonator it traverses. The atom is the quanton; the relevant photonic degrees of freedom constitute the marker. The wave trains are shown at three different instants: before entering the resonators; after emerging from them, but before reaching the slits; after having been diffracted at the slits. The state vector $|\Psi\rangle$ of equation (10) refers to the latter instant.

resonators. The resonators are initially empty, and matters are arranged such that each atom emits — with certainty — a photon into a privileged mode of the resonator it traverses.

This is then the situation: The atom is the quanton, and its center-of-mass degrees of freedom are the ones relevant for the interferometer. The marker is made up by the photonic degrees of freedom of the selected resonator modes. The quanton states $|S_1\rangle$ and $|S_2\rangle$ are the diffracted de Broglie waves, as indicated in figure 1, and

$$\begin{aligned}
 |M_1\rangle &\equiv \text{“one photon in the 1}^{\text{st}} \text{ resonator,} \\
 &\quad \text{none in the 2}^{\text{nd}},\text{”} \\
 |M_2\rangle &\equiv \text{“one photon in the 2}^{\text{nd}} \text{ resonator,} \\
 &\quad \text{none in the 1}^{\text{st}},\text{”}
 \end{aligned}
 \tag{15}$$

specifies the significance of the marker states. The value of the interferometric phase ϕ of equation (5) is determined by the site at which the atom hits the screen.

In the second example of figure 2 we have a Mach-Zehnder interferometer for light, operated in a one-photon-at-a-time fashion. The entering photon encounters a

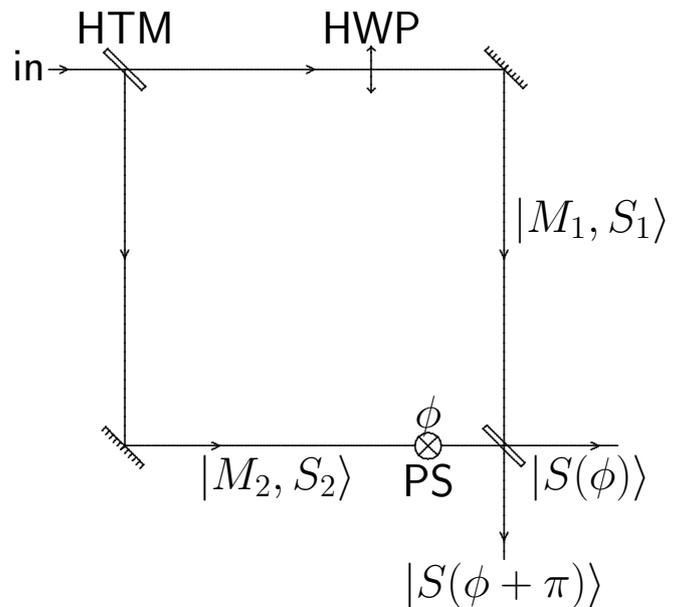


FIG. 2. A polarized photon enters a Mach-Zehnder interferometer through a half-transparent mirror HTM. The transmitted amplitude passes through a half-wave plate HWP which changes the polarization to an orthogonal one. The quanton is made up by the orbital degrees of freedom of the photon, the polarization is the marker. A phase shifter PS and a second HTM combine the quanton states so that the superpositions $|S(\phi)\rangle$ and $|S(\phi + \pi)\rangle$ of equation (5) emerge at the two output ports.

half-transparent mirror (HTM); the two resulting parts of the photon’s orbital amplitude are the quanton states $|S_1\rangle$ and $|S_2\rangle$. A phase shifter (PS) and a second HTM make the superposition $|S(\phi)\rangle$ leave at the symmetric output port and the orthogonal one, that is: $|S(\phi + \pi)\rangle$, at the asymmetric one.

The polarization of the photon is used for WW marking. For this purpose, all entering photons are polarized horizontally, say. A half-wave plate (HWP) in the $|S_1\rangle$ arm of the interferometer turns the polarization from horizontal to vertical. Here, the photon’s polarization degree-of-freedom constitutes the marker in accordance with

$$\begin{aligned}
 |M_1\rangle &\equiv \text{“vertically polarized,”} \\
 |M_2\rangle &\equiv \text{“horizontally polarized.”}
 \end{aligned}
 \tag{16}$$

It is, of course, necessary to detect the photons at the output together with their polarization.

A Mach-Zehnder interferometer is also employed in the third example (figure 3), which is a variant of problem 9-6 in L. E. Ballentine’s textbook [7]. Here we begin with a

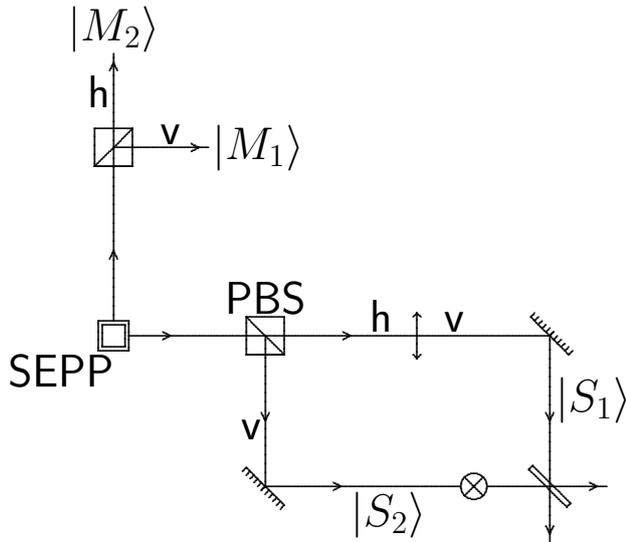


FIG. 3. A source of entangled photon pairs SEPP emits polarization-entangled EPR pairs. One photon, the quanton, propagates to the right and enters a Mach-Zehnder interferometer through a polarizing beam splitter PBS. Vertical polarization (v) is reflected, horizontal polarization (h) is transmitted. The latter is changed to vertical also by a HWP. The second photon of the pair, whose polarization is the marker, propagates upwards and is detected behind another PBS.

source of entangled photon pairs (SEPP).¹⁴ It emits EPR pairs of photons going to the right or going up, so that one photon is polarized horizontally and the other one vertically. The emitted pairs are in a state such as

$$\frac{1}{\sqrt{2}}(|h_{\rightarrow}, v_{\uparrow}\rangle + |v_{\rightarrow}, h_{\uparrow}\rangle), \quad (17)$$

where, for instance, h_{\rightarrow} means “horizontally polarized photons goes to the right.”

Of each pair, the photon that goes to the right is the quanton. It enters a Mach-Zehnder interferometer with a polarizing beam splitter (PBS) at the entrance, which reflects vertically polarized photons and transmits horizontally polarized ones. The transmitted amplitude passes through a HWP, so that the quanton photon is surely polarized vertically when it reaches the PS and HTM that probe for $|S(\phi)\rangle$ and $|S(\phi + \pi)\rangle$.

The polarization of the up-going photon is used for WW marking. The states $|M_1\rangle$ and $|M_2\rangle$ stand for vertical and horizontal polarization, respectively, and can be probed by ordinary photodetection behind another PBS.

The three examples exhibit three different methods for WW marking. In figure 1 another physical system (the

resonators with their photon modes) is used. By contrast, in figure 2 the same physical object — the photon — serves both as quanton (orbital motion) and as marker (polarization). In these two examples, the WW marking is done during the passage of the quanton through the apparatus, and one could operate the interferometer without WW marking. Not so in the third example, where the quanton and the marker are created jointly (by the SEPP) and are in the entangled quanton-and-marker state to begin with. As a consequence, the way through the interferometer can be known even before the quanton reaches the entry port. In other words, the way is *predictable* in the set-up of figure 3, and therefore this set-up exemplifies the situation considered by G. Jaeger, A. Shimony, and L. Vaidman in reference [38].

The thought experiment of figure 1 will most likely never be realized, but there are other, more realistic, schemes in which a privileged photon mode of a resonator is used for WW marking. In particular, Ramsey interferometers for atoms in Rydberg states are well suited for this purpose. One can either use a resonant interaction, as in the proposal of reference [39], or a dispersive interaction, as in the experiment reported in reference [40]. The systematic loss of the fringe visibility has been observed in this experiment, but WW information has not been extracted as yet.

P. G. Kwiat and P. D. D. Schwindt of Los Alamos National Laboratory have recently built the interferometer of figure 2 [41]. They have not only succeeded in demonstrating the systematic dependence of the fringe visibility on the parameters, but have also obtained WW information from the final polarization state (in the quantitative manner discussed in section 10). This two-fold challenge has also been met by S. Dürr, T. Nonn, and G. Rempe of the University of Konstanz who have built an atom interferometer that uses hyperfine sublevels for the WW marking [42,43].

Finally, concerning figure 3, this should be a relatively easy experiment if a SEPP is at hand. An actual realization could be very rewarding, in particular because the feature of “late choice” (see section 7) can be incorporated rather simply.

5. Entanglement and correlations

As long as there is no further interaction between the quanton and the WW marker, the statistical operator ρ_Q of (13) obtains independently of the history of the WW marker after the quanton-marker interaction. The statistical properties of all measurements that could be performed on the quanton are correctly predicted by this ρ_Q . Accordingly, this statistical operator characterizes the state of the quanton after its interaction with the WW marker just like $\rho_Q^{(0)}$ of (4) did before the interaction.

Likewise we have a statistical operator for the WW marker,

¹⁴Such sources are actually available, see reference [35]; the recent teleportation experiments [36,37] made use of SEPPs.

$$\rho_M = \text{tr}_Q \{P\} = \frac{1}{2}(|M_1\rangle\langle M_1| + |M_2\rangle\langle M_2|), \quad (18)$$

which characterizes the marker state after the interaction. The (direct) product of ρ_M and ρ_Q ,

$$\begin{aligned} \rho_M \rho_Q = \frac{1}{4} & (|M_1, S_1\rangle\langle M_1, S_1| + |M_2, S_2\rangle\langle M_2, S_2| \\ & + |M_1, S_2\rangle\langle M_1, S_2| + |M_2, S_1\rangle\langle M_2, S_1|), \end{aligned} \quad (19)$$

differs from P of (11). This is no surprise, of course, because the quanton and the marker are *entangled* (German: *verschränkt* or *verheddert*, as coined by E. Schrödinger [8]) if P of (11) applies, but they would not be entangled if the product $\rho_M \rho_Q$ represented their statistical properties. As a consequence of this entanglement, joint probabilities of measurements on both the quanton and the marker are not simply equal to the product of the individual probabilities. For instance, the probability for finding the quanton in state $|S_1\rangle$ is 50%, and this is also the probability for finding the WW marker in state $|M_2\rangle$; but the joint probability for finding the quanton in state $|S_1\rangle$ and also the marker in state $|M_2\rangle$ is 0% — not 25% as would be the case if (19) were the statistical operator. This exemplifies the general situation:

Entanglement between different degrees of freedom results in *correlations* between the results of measurements on the subsystems. (20)

And vice versa: if measurement results are correlated, then the subsystems must be entangled.¹⁵

We note that the entanglement of two subsystems can be of a purely classical nature. For the quanton-and-marker system this would be the case, for example, if its statistical operator were given by

$$\frac{1}{2}(|M_1, S_1\rangle\langle M_1, S_1| + |M_2, S_2\rangle\langle M_2, S_2|), \quad (21)$$

where the cross terms of (11) are missing. The corresponding situation would be this one: Either the state $|M_1, S_1\rangle$ is realized or the state $|M_2, S_2\rangle$ (with equal probability), but we don't know which one. In other words: (21) represents a classical mixture of disentangled states. Inasmuch as statement (12) is as true for (21) as it is for (11), the statistical operator (21) surely describes an entangled quanton-and-marker system. But, owing to the absence of the cross terms in (21), the correlations are here not of the quantum-mechanical EPR type.

¹⁵There may also be correlations of a different kind for measurements on a single degree of freedom in the sense that the expectation values of two observables A and B and of their symmetrized product $AB + BA$ are such that $\langle AB + BA \rangle$ does not equal twice the product of $\langle A \rangle$ and $\langle B \rangle$. Such correlations are ubiquitous, but they are clearly of quite a different nature than the EPR ones.

6. Mixtures, blends, and as-if-realities

6.1. Blends correspond to as-if-realities

If the interferometer is operated such that only one way is realized with certainty, then one has $\rho_Q = |S_1\rangle\langle S_1|$ or $\rho_Q = |S_2\rangle\langle S_2|$, of course. The statistical operator ρ_Q of (13) is a 50:50 blend of these cases. Does this mean that the quanton is either in state $|S_1\rangle$ or in state $|S_2\rangle$ but we simply don't know in which one? This interpretation suggests itself and does not lead to inconsistencies. One must, however, be aware that it represents only an *as-if-reality*. The statistical predictions resulting from ρ_Q of (13) are such that it appears *as if* the quanton were in $|S_1\rangle$ or $|S_2\rangle$. Equally well one can regard this ρ_Q as a 50:50 blend of another pair of orthonormal states, such as

$$\begin{aligned} |\tilde{S}_1\rangle &= |S_1\rangle \cos \vartheta + |S_2\rangle e^{i\varphi} \sin \vartheta, \\ |\tilde{S}_2\rangle &= |S_2\rangle \cos \vartheta - |S_1\rangle e^{-i\varphi} \sin \vartheta \end{aligned} \quad (22)$$

because

$$\rho_Q = \frac{1}{2}(|\tilde{S}_1\rangle\langle\tilde{S}_1| + |\tilde{S}_2\rangle\langle\tilde{S}_2|) \quad (23)$$

holds not only for the $\vartheta = 0$ case of equation (13) but for all values of the parameters ϑ and φ . Consequently, it is also *as if* the quanton were either in state $|\tilde{S}_1\rangle$ or in $|\tilde{S}_2\rangle$, but we don't know in which one.

In equations (13) and (23) we have a simple example for the general observation that a statistical operator can be mingled from (projectors to) pure states in a plethora of different choices, with a corresponding abundance of equally good as-if-realities. Following G. Süßmann [9] we say that there are many *blends* (German: *Gemenge*), exemplified by the right-hand sides of (13) and (23), that make up one and the same *mixture* (German: *Gemisch*), exemplified by the identical left-hand sides.

In general terms, any decomposition of a statistical operator into a convex sum of projectors (the statistical operators of pure states),

$$\rho = \sum_k w_k \rho_k \quad (24a)$$

with

$$\rho_k^2 = \rho_k, \quad \text{tr}\{\rho_k\} = 1, \quad w_k > 0, \quad \sum_k w_k = 1, \quad (24b)$$

identifies one of the blends that compose the given mixture ρ . Each blend is associated with an interpretation in terms of an as-if-reality: “The system is in one of the states ρ_k , with statistical weights w_k , but we don't know in which one.” which yields a consistent picture of the phenomena — nothing more, and nothing less.

It is not necessary that the ρ_k 's are mutually orthogonal in equations (24). If they happen to be, then we

are dealing with (one of) the spectral decomposition(s) of the mixture ρ . Such blends are mathematically particular, but not physically.

Further we note that the sums in (24) could be integrals. For example,

$$\rho_Q = \int_0^{\pi/2} d\vartheta \sin(2\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi} |\tilde{S}_1\rangle\langle\tilde{S}_1| \quad (25)$$

blends ρ_Q from all possible projectors $|\tilde{S}_1\rangle\langle\tilde{S}_1|$ with statistical weights that are uniform in $\cos(2\vartheta)$ and φ .

6.2. All blends are equal

Repeated measurements on quantons prepared in the same mixture cannot, by any means, distinguish one blend, or one as-if-reality, from the other. Whereas it is a matter of personal taste or preconceptions whether one regards this impossibility as an indication that quantum mechanics does not provide for a complete description, it is certainly not a matter of opinion that this impossibility is fundamental.¹⁶ For, if one could tell one blend from the other (hypothetical nonlinear additions to the Schrödinger equation would enable one to achieve this) then one could send signals at arbitrary speed [44] and get in violent conflict with Einsteinian causality.

The marker-quanton entanglement in (10) and (11) and, in particular, the resulting equivalence stated in (12) invite the surmise that the $\vartheta = 0$ blend (13) is, in some sense, more physical than the $\vartheta \neq 0$ ones of (23), irrespective of what is said in the preceding paragraph. After all, couldn't it be that, paraphrasing Orwell, all as-if-realities are equal but some are more equal than others? No, certainly not, because we can cast the state vector (10) into the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\tilde{M}_1, \tilde{S}_1\rangle + |\tilde{M}_2, \tilde{S}_2\rangle) \quad (26)$$

where

$$\begin{aligned} |\tilde{M}_1\rangle &= |M_1\rangle \cos \vartheta + |M_2\rangle e^{-i\varphi} \sin \vartheta, \\ |\tilde{M}_2\rangle &= |M_2\rangle \cos \vartheta - |M_1\rangle e^{i\varphi} \sin \vartheta \end{aligned} \quad (27)$$

are the superpositions of the marker states that correspond to (22). Accordingly, the correlations between $|\tilde{M}_1\rangle$ and $|\tilde{S}_1\rangle$ as well as $|\tilde{M}_2\rangle$ and $|\tilde{S}_2\rangle$ are equally strong for all values of ϑ and φ ; the $\vartheta = 0$ case of (12) is not distinguished at all. Indeed, we can generalize this statement about EPR correlations:

¹⁶L. E. Ballentine discusses, in section 9-4 of his textbook [7], an instructive example of two equivalent as-if-realities and reports a historical incident that illustrates well the confusion that may arise when the as-if character is not appreciated.

Irrespective of the chosen setting of the parameters ϑ and φ , knowing that the WW marker is in state $|\tilde{M}_1\rangle$ or $|\tilde{M}_2\rangle$ is tantamount to knowing that the quanton is in state $|\tilde{S}_1\rangle$ or $|\tilde{S}_2\rangle$, respectively. (28)

As a consequence, in repeated experiments that are all accounted for by the statistical operator of (11), the experimenter can choose, at his discretion, how he wants to analyze the data. Each quanton can be identified as the member of either the subensemble specified by $|\tilde{S}_1\rangle$ or the one specified by $|\tilde{S}_2\rangle$, whereby the values of ϑ and φ can vary from one quanton to the next — at least in principle, if not in practice. This sorting of quantons into subensembles is done with the aid of a measurement of the marker state — that is: a certain “reading of the marker” — a measurement that discriminates between $|\tilde{M}_1\rangle$ and $|\tilde{M}_2\rangle$, and does not involve an observation of the quanton itself.

Let us take a look at the interference pattern from the point of view offered by the as-if-reality that goes with the blend of (23). To avoid ambiguities we restrict ϑ and φ to the ranges $0 \leq \vartheta \leq \pi/4$, $0 \leq \varphi < 2\pi$, which are large enough to cover all possibilities.¹⁷ We have

$$p(\phi) = \frac{1}{2} |\langle S(\phi) | \tilde{S}_1 \rangle|^2 + \frac{1}{2} |\langle S(\phi) | \tilde{S}_2 \rangle|^2 = \frac{1}{2} \quad (29)$$

with

$$|\langle S(\phi) | \tilde{S}_1 \rangle|^2 = \frac{1}{2} [1 + \sin(2\vartheta) \cos(\phi - \varphi)] \quad (30a)$$

and

$$|\langle S(\phi) | \tilde{S}_2 \rangle|^2 = \frac{1}{2} [1 - \sin(2\vartheta) \cos(\phi - \varphi)]. \quad (30b)$$

Thus it is *as if* $p(\phi) = \frac{1}{2}$ obtains because there are “really” two patterns, each with a fringe visibility of $\mathcal{V} = \sin(2\vartheta)$ and the fringes shifted by φ and $\varphi + \pi$, respectively, as compared with the pattern of (6). In the fringeless sum of (29), the crests of (30a) meet the troughs of (30b) and vice versa.

7. Quantum erasure

We are now prepared for discussing the kind of sorting that goes by the name of quantum erasure. Consider the following scenario. A quanton is sent through the interferometer equipped with the WW detection device, so that (11) gives a correct account of the statistical predictions of the experiment. The interferometer phase ϕ is set to a chosen value, and the observable

¹⁷This ϑ range is only half of the range of integration in (25) because the replacements $\vartheta \rightarrow \pi/2 - \vartheta$, $\varphi \rightarrow \varphi \pm \pi$ would just exchange $|\tilde{S}_1\rangle\langle\tilde{S}_1|$ and $|\tilde{S}_2\rangle\langle\tilde{S}_2|$.

$$O(\phi) = |S(\phi)\rangle\langle S(\phi)| \quad (31)$$

is measured. In accordance with (14) the measurement results 0 and 1 are equally probable. Another measurement determines whether the WW marker is in state $|M_1\rangle$ or in $|M_2\rangle$. We adopt this color coding: Finding $|M_1\rangle$ identifies the quanton as a member of the blue subensemble, and the quanton belongs to the yellow subensemble if $|M_2\rangle$ is found. We have $\vartheta = 0$ here, and equations (30) tell us that in both subensembles the eigenvalues of $O(\phi)$ occur with the same frequency. In other words: The blue quantons exhibit no interference fringes, nor do the yellow ones.

Of course, this is not surprising, but expected because the blue quantons are those which are known to have taken the way through slit 1 and the yellow ones the way through slit 2. We are just repeating what is stated in (12).

Alternatively, the experimenter can decide to make another measurement on the marker, one that distinguishes $|\widetilde{M}_1\rangle$ from $|\widetilde{M}_2\rangle$ of (27) for a setting of ϑ and φ . In particular, the choice $\vartheta = \pi/4$, $\varphi = 0$ leads to the distinction of

$$|M_+\rangle = \frac{1}{\sqrt{2}}(|M_1\rangle + |M_2\rangle) \quad (32a)$$

and

$$|M_-\rangle = \frac{1}{\sqrt{2}}(|M_2\rangle - |M_1\rangle). \quad (32b)$$

In the set-up of figure 3, for example, such a $|M_+\rangle/|M_-\rangle$ distinction is done by turning the polarization-sensitive beam splitter in the marker part by 45° .

We extend the color coding: Finding the marker state $|M_+\rangle$ puts the quanton into the red subensemble, and finding $|M_-\rangle$ puts it into the green one. According to (30a), the red quantons exhibit the original interference pattern of (6) with perfect fringe visibility, and the green quantons yield the antifrings.

Again, this shouldn't come as a surprise because we are simply facing a special case of (28), namely this one: Each red quanton is known to be in the state

$$|S_+\rangle = \frac{1}{\sqrt{2}}(|S_1\rangle + |S_2\rangle) \quad (33a)$$

and each green one is in

$$|S_-\rangle = \frac{1}{\sqrt{2}}(|S_2\rangle - |S_1\rangle). \quad (33b)$$

For the red quantons and the green ones it is unknown through which slit they passed — in fact not just unknown, but rather unknowable. As a consequence, the alternatives interfere.

As soon as the $|M_+\rangle/|M_-\rangle$ -distinguishing measurement has been made, it is no longer possible to make

the $|M_1\rangle/|M_2\rangle$ distinction, because these two measurements are incompatible. There is hardly a point in looking for $|M_1\rangle$ or $|M_2\rangle$ when it is already known that the marker is in state $|M_+\rangle$, say, since we learn nothing about the state of the marker prior to the $|M_+\rangle/|M_-\rangle$ -distinguishing measurement. In a manner of speaking, the $|M_+\rangle/|M_-\rangle$ -distinguishing measurement has erased whatever WW information was there before. This circumstance gave the name *quantum erasure* (QE) to the red/green sorting [45,34,46]. We shall therefore speak of the QE measurement when the $|M_+\rangle/|M_-\rangle$ distinction is made, and of the WW measurement in case of the $|M_1\rangle/|M_2\rangle$ distinction.

Inasmuch as WW information is particlelike and interference fringes are evidence for wavelike properties, one could also say that the WW sorting processes the data such that a particle experiment is effectively carried out, whereas the QE sorting amounts to doing a wave experiment. The freedom of deciding at a very late stage, possibly after the performance of the interferometric $O(\phi)$ measurement, whether the quanton in question will be part of the particle experiment or of the wave experiment, demonstrates that QE offers a particular realization of experiments with *delayed choice*, which were first discussed by C. F. von Weizsäcker in 1941 [47] and became popular from 1978 on as a result of J. A. Wheeler's efforts [48].

In the set-up of figure 3, for example, the marker photon, which propagates upwards from the SEPP, can be sent on a detour through a long fiber. The experimenter can then first check at which output port the quanton photon emerges from the Mach-Zehnder interferometer, and then choose one of the sortings. This late-choice aspect of QE is especially intriguing.

For each quanton traversing the interferometer two measurements are made, the measurement of $O(\phi)$ on the quanton and the WW or the QE measurement on the WW marker. Does the joint probability of getting $O(\phi) = 1$ and ending up in the red ensemble, say, depend on the order in which the two measurements are performed? Surely it doesn't because the two measurements refer to different degrees of freedom; the corresponding operators commute. Therefore, the experimenter can first measure $O(\phi)$ and then decide whether the quanton in question should get a red/green label (QE sorting) or a blue/yellow label (WW sorting). This freedom enables the experimenter to sort the data in fancy manners if he likes to do so. For instance, quantons that hit one half of the screen of a two-slit interferometer could be labeled by blue and yellow, and those hitting the other half by red and green. In view of (28), it is clear that much more complicated sorting schemes are conceivable.

8. Post festum

No one doubts that these joint probabilities are independent of the order in which the $O(\phi)$ measurement

and the sorting measurement are done for each individual quanton. Nevertheless, objections have been raised against the statement that the temporal order is truly without significance, and thus against the freedom of a late choice between the WW and the QE sorting. In particular, it has been argued — both in private and in public; recently and eloquently by U. Morhoff [49] — that the reading of the WW marker must be done before the $O(\phi)$ measurement is performed. Or, put differently, one has to sort the quanton into one of the color-coded subensembles first and then determine its contribution to the interference pattern. When $O(\phi)$ is measured first, so the critics assert, nothing can be learned from a subsequent, *post festum* reading of the WW marker.

I disagree [50] because this critique is at odds with the objective nature of the EPR correlations (28) that link quanton states to corresponding marker states.

Consider the situation in which $O(\phi)$ of (31) has been measured and the eigenvalue 1 found. This characterizes a subensemble, for which the statistical operator of the marker is

$$\rho_M = \frac{\text{tr}_Q \{O(\phi)PO(\phi)\}}{\text{tr}_M \text{tr}_Q \{O(\phi)PO(\phi)\}} = |M(\phi)\rangle\langle M(\phi)| \quad (34)$$

with

$$|M(\phi)\rangle = \frac{1}{\sqrt{2}}(|M_1\rangle + |M_2\rangle e^{-i\phi}). \quad (35)$$

The conditional probabilities for the outcomes of the WW measurement are then

$$\begin{aligned} \text{blue:} & \quad |\langle M_1|M(\phi)\rangle|^2 = \frac{1}{2}, \\ \text{yellow:} & \quad |\langle M_2|M(\phi)\rangle|^2 = \frac{1}{2}, \end{aligned} \quad (36)$$

and those of the QE measurement are

$$\begin{aligned} \text{red:} & \quad |\langle M_+|M(\phi)\rangle|^2 = \frac{1}{2}(1 + \cos \phi), \\ \text{green:} & \quad |\langle M_-|M(\phi)\rangle|^2 = \frac{1}{2}(1 - \cos \phi). \end{aligned} \quad (37)$$

So, this subensemble contains equal numbers of blue and yellow quantons but the relative frequency of red and green ones will be biased if $\cos \phi \neq 0$.

Once more, there is no surprise. As soon as the eigenvalue 1 of $O(\phi)$ has been found, it is known that the quanton is in state $|S(\phi)\rangle$ which, according to (28), is tantamount to knowing that the marker is in state $|M(\phi)\rangle$, as is confirmed by (34). And the probabilities that the quanton is blue, yellow, red, or green are the ones stated above.

The said critics do not dispute these facts, but they give them a twist to arrive at their central argument, which is essentially as follows. Having found the eigenvalue 1 of $O(\phi)$, the statistical operator of the joint quanton-marker system is no longer given by P of (11)

but rather by the one of the subensemble characterized by the measurement result, that is

$$\begin{aligned} P^{(\phi)} &= |M(\phi), S(\phi)\rangle\langle M(\phi), S(\phi)| \\ &= |\Psi^{(\phi)}\rangle\langle\Psi^{(\phi)}| = \rho_M \rho_Q. \end{aligned} \quad (38)$$

As indicated, this factors into a marker part with ρ_M of (34) and a quanton part with ρ_Q equal to $O(\phi)$ of (31), and therefore there is no entanglement between the marker and the quanton in $P^{(\phi)}$. In other words: The equivalence stated in (28) is not true for $P^{(\phi)}$, only for P . And so, the critics conclude, after the $O(\phi)$ measurement has been performed, measurements on the marker will no longer tell us anything about the state of the quanton.

Superficially it appears that a valid objection has been raised. Actually, however, the critics are missing the point.

The purpose of the WW marker is to *store* information about the state of the quanton. Manipulations of, such as measurements on, the quanton do not affect what is stored in the WW marker. Perhaps the following scene from the lab illustrates the issue. Student Alice has the job of making measurements on the quanton, student Bob reads the WW marker. Two experimental situations, E1 and E2, are of interest:

E1: Bob finds the marker in state $|\widetilde{M}_1\rangle$ and tells Alice about it. Then Alice makes a $|\widetilde{S}_1\rangle/|\widetilde{S}_2\rangle$ -distinguishing measurement on the quanton as a test of the prediction that the quanton is in state $|\widetilde{S}_1\rangle$. Indeed, this is what she will always find under the circumstances stated, and so Alice confirms what is said in (28).

E2: Bob finds the marker in state $|\widetilde{M}_1\rangle$ and tells Alice about it. But before the message arrives, Alice measures $O(\phi)$. Upon Bob's request to confirm that the quanton is in state $|\widetilde{S}_1\rangle$, she replies: Sorry, too late. — In many repetitions of these circumstances, Alice will confirm that the results of the $O(\phi)$ measurement are statistically consistent with Bob's findings about the marker state.

E2 has occurred and the supervisor turns to Alice with the question: “Unfortunately Bob's message has arrived after your $O(\phi)$ measurement. If you had made the $|\widetilde{S}_1\rangle/|\widetilde{S}_2\rangle$ distinction, as in E1, instead of measuring $O(\phi)$, what would you have found?” Undoubtedly she answers: “I would have found $|\widetilde{S}_1\rangle$, of course!” because this is the logical implication of the empirical experience gained in E1. Naturally, the supervisor is pleased.

The day after, Alice falls ill and Chuck takes her place. At the end of the day the supervisor comes to the lab and asks Chuck the very same question to which Alice had responded so pleasingly. Chuck, however, gives a different answer; he says: “Textbooks on quantum mechanics warn against making statements about the hypothetical outcome of measurements that haven't been performed

and can no longer be made. Therefore, I'd say there is no sensible answer to your question." The supervisor isn't happy at all with this reply, because Chuck ignores the lesson of E1.

Very often, statements about the outcomes of measurements that could have been performed but have not been made actually, do not make much sense — very often, but not always. Alice is on safe ground here because of her E1 experience. Her answer is not imprudent, but demonstrates a thorough understanding of the characteristics of the experiment.

Does Alice's answer "I would have found $|\tilde{S}_1\rangle$, of course!" depend on the time when Bob makes his measurement? No, it doesn't because the correlations of (28) are reciprocal, so that the roles of Alice and Bob can be interchanged in E1. According to the post-festum critics, however, Alice's answer should depend on the timing. When her measurement is first, she should give Chuck's answer. The critics' fallacy is thus brought to light: Their reasoning is at variance with the empirical reality of the EPR correlations stated in (28) and confirmed by E1. Case closed.

The critics are led astray by regarding the state reduction that turns P of (11) into $P^{(\phi)}$ of (38) as a physical process, not as the mental process it is. This point of view necessarily requires that the original state vector $|\Psi\rangle$ of (10) as well as the reduced one, $|\Psi^{(\phi)}\rangle$ of (38), are regarded as real physical objects, rather than as the book-keeping devices that they are. We recall the recommended minimalistic interpretation: The state vector $|\Psi\rangle$ serves the sole purpose of summarizing concisely our knowledge about the entangled quanton-and-marker system; in conjunction with the known dynamics, it enables us to make correct predictions about the statistical properties of future measurements. Whoever endows $|\Psi\rangle$ with more meaning than that ...

The notion of *state reduction* that just came up is nothing mysterious. It is dictated by the rules of correct book-keeping. We begin with a statistical operator P that refers to a certain ensemble (here: of quantons entangled with WW markers) and summarizes what we know about it. Then a measurement result (here: "O(ϕ) equals 1") is used to identify a subensemble. Probabilistic predictions concerning this subensemble cannot be based on the original P, but must rely on a suitably refined P_{sub} (here: $P^{(\phi)}$) that accounts for the defining properties of the subensemble. In other words: P_{sub} yields the correct *conditional probabilities*, conditioned on the said measurement result. The transition $P \rightarrow P_{\text{sub}}$ is the corresponding state reduction. Clearly, it is not a physical process, but a mental one that simply reflects the change in our knowledge about the system. In addition, state reduction is not a specialty of quantum mechanics; it is a technical device of all statistical theories: "when I toss a coin the 50–50 probability distribution changes abruptly if I look at the outcome" (N. G. van Kampen [19]).

9. What does a quantum eraser erase?

9.1. More realistic which-way markers

The analysis of sections 2 and 7 owes its simplicity to the idealization that the pure states $|M_1\rangle$ and $|M_2\rangle$ suffice for an appropriate description of the WW marker. We lift this restriction now and suppose more generally that the WW marker is initially prepared in a state characterized by the statistical operator $\rho_M^{(0)}$. The quanton is initially in the state $|\psi\rangle$ of (2), so that $\rho_Q^{(0)}$ of (4) is the initial statistical operator, for which we now write more compactly

$$\rho_Q^{(0)} = \frac{1}{2}(\sigma^\dagger\sigma + \sigma\sigma^\dagger + \sigma + \sigma^\dagger) \quad (39)$$

where

$$\sigma = |S_2\rangle\langle S_1|, \quad \sigma^\dagger = |S_1\rangle\langle S_2| \quad (40)$$

are obvious analogs of spin-flip operators.

The net effect of the interaction that creates the entanglement between marker and quanton is accounted for by two unitary operators, U_1 and U_2 , one for each way, that is:

$$\rho_M^{(0)} \rightarrow \begin{cases} U_1^\dagger \rho_M^{(0)} U_1 \equiv \rho_M^{(1)} & \text{for way 1,} \\ U_2^\dagger \rho_M^{(0)} U_2 \equiv \rho_M^{(2)} & \text{for way 2.} \end{cases} \quad (41)$$

Inasmuch as $\sigma^\dagger\sigma$ and $\sigma\sigma^\dagger$ are the projectors that select the ways, the unitary operator that transforms the joint marker-quanton system is

$$U_{M\&Q} = \sigma^\dagger\sigma U_1 + \sigma\sigma^\dagger U_2, \quad (42)$$

and the entangled state

$$P = U_{M\&Q}^\dagger \rho_M^{(0)} \rho_Q^{(0)} U_{M\&Q} \quad (43)$$

results from the marker-quanton interaction. Its explicit form is

$$P = \frac{1}{2} \left(\underbrace{\sigma^\dagger\sigma \rho_M^{(1)}}_{\text{slit 1}} + \underbrace{\sigma\sigma^\dagger \rho_M^{(2)}}_{\text{slit 2}} + \underbrace{\sigma\tilde{\rho}_M + \sigma^\dagger\tilde{\rho}_M^\dagger}_{\text{cross terms}} \right) \quad (44)$$

where the single-slit contributions involve the marker states $\rho_M^{(1)}$ and $\rho_M^{(2)}$ corresponding to them, and the cross terms contain

$$\tilde{\rho}_M = U_2^\dagger \rho_M^{(0)} U_1 \quad (45)$$

and its adjoint. Of course, the statistical operator of (11) is a particular realization of (44); for example, the choice $\rho_M^{(0)} = |M_1\rangle\langle M_1|$ in conjunction with $U_1 = 1$ and U_2 such that $\langle M_1|U_2 = \langle M_2|$ turns (44) into (11).

Partial traces produce the statistical operators for the WW marker,

$$\rho_M = \text{tr}_Q \{P\} = \frac{1}{2}(\rho_M^{(1)} + \rho_M^{(2)}), \quad (46)$$

and for the quanton

$$\rho_Q = \text{tr}_M \{P\} = \frac{1}{2}(\sigma^\dagger \sigma + \sigma \sigma^\dagger + \mathcal{C} \sigma + \mathcal{C}^* \sigma^\dagger). \quad (47)$$

The significance of the complex number

$$\mathcal{C} = \text{tr}_M \{\tilde{\rho}_M\} = \text{tr}_M \{U_2^\dagger \rho_M^{(0)} U_1\} \quad (48)$$

is revealed by a glance at the resulting interference pattern,

$$p(\phi) = \text{tr}_Q \{O(\phi) \rho_Q\} = \frac{1}{2}[1 + \text{Re}(e^{-i\phi} \mathcal{C})]. \quad (49)$$

Inasmuch as

$$\left. \begin{array}{l} p_{\max} \\ p_{\min} \end{array} \right\} = \frac{1}{2}(1 \pm |\mathcal{C}|) \quad (50)$$

are the extreme values of $p(\phi)$, the fringe visibility \mathcal{V} equals the modulus of \mathcal{C} ,

$$\mathcal{V} = |\mathcal{C}|, \quad (51)$$

and the argument of \mathcal{C} determines the location of the crests and troughs.

A well functioning WW detection device is such that $\rho_M^{(1)}$ and $\rho_M^{(2)}$ can be kept apart. (The relevant numerical measure is introduced in section 10; here we'll get around without technicalities of this kind.) It is clear that both the initial marker state $\rho_M^{(0)}$ and the unitary operators U_1, U_2 must be chosen judiciously to achieve well distinguishable final marker states.

A particularly unfortunate choice is exemplified by

$$\rho_M^{(0)} = \frac{1}{2}(|M_1\rangle\langle M_1| + |M_2\rangle\langle M_2|) \quad (52a)$$

in conjunction with

$$\begin{aligned} U_1 &= 1, \\ U_2 &= |M_1\rangle\langle M_2| + |M_2\rangle\langle M_1| + \dots, \end{aligned} \quad (52b)$$

where the ellipsis indicates those irrelevant pieces of U_2 that act on marker states orthogonal to both $|M_1\rangle$ and $|M_2\rangle$. Here one gets

$$\rho_M^{(1)} = \rho_M^{(2)} = \rho_M^{(0)} \quad (52c)$$

so that no WW information is available, and

$$\tilde{\rho}_M = \frac{1}{2}(|M_2\rangle\langle M_1| + |M_1\rangle\langle M_2|), \quad (52d)$$

implying

$$\mathcal{C} = 0, \mathcal{V} = 0, \quad (52e)$$

is bad news too: No fringes.

9.2. The as-if-reality of quantum erasure

This simple example illustrates graphically that the lack of WW information does not ensure good fringe visibility, nor does a fringeless pattern indicate that WW information has become available. The implication works only in the opposite directions:

$$\begin{array}{l} \text{Full fringe visibility precludes any WW} \\ \text{information, and the acquisition of com-} \\ \text{plete WW knowledge enforces the disap-} \\ \text{pearance of interference fringes.} \end{array} \quad (53)$$

Intermediate situations are the subject of section 10.

In section 7 we mentioned that the data sorting called quantum erasure got its name because in the course of performing QE the WW information is lost — it is “erased.” Now we are facing a new situation in which the interference fringes are gone, but no WW information has been gained to compensate for the loss. Is a data sorting of the QE type still possible although there is no WW information that could be erased?

Yes, QE is still possible. We justify this affirmative answer in the general context of (44). To perform QE one makes a measurement on the marker that distinguishes the states $|\alpha_\nu\rangle$ whose defining property is

$$U_1 |\alpha_\nu\rangle = U_2 |\alpha_\nu\rangle e^{i\alpha_\nu}. \quad (54)$$

These $|\alpha_\nu\rangle$'s are the eigenstates of the unitary operator $U_2^\dagger U_1$ and the phase factors $\exp(i\alpha_\nu)$ are the respective eigenvalues. When the marker is found in the state $|\alpha_\nu\rangle$, the corresponding subensemble of quantons is characterized by

$$\rho_Q^{(\nu)} = \frac{1}{2}(\sigma^\dagger \sigma + \sigma \sigma^\dagger + e^{i\alpha_\nu} \sigma + e^{-i\alpha_\nu} \sigma^\dagger). \quad (55)$$

Since this is essentially ρ_Q of (47) with $\mathcal{C} \rightarrow \exp(i\alpha_\nu)$, the interference pattern of this subensemble is [cf. (49)]

$$p^{(\nu)}(\phi) = \frac{1}{2}[1 + \cos(\phi - \alpha_\nu)]. \quad (56)$$

So, when the quantons are sorted according to the result of the $|\alpha_\nu\rangle$ -distinguishing measurement, then each subensemble shows an interference pattern with unit fringe visibility. Therefore, the said measurement is a QE measurement. This analysis does not make use of any special properties of U_1 and U_2 , and makes no reference at all to $\rho_M^{(0)}$. Consequently, it is indeed true that QE can be performed even in the unfortunate case specified by (52).

We have thus seen quite explicitly that QE is possible even if there is no WW information to be erased. If it's not the WW information, then what does a quantum eraser erase?

The answer is based on the observation that there is an as-if-reality to ρ_Q of (47) that goes with the $\rho_Q^{(\nu)}$'s of (55):

$$\rho_Q = \sum_{\nu} w^{(\nu)} \rho_Q^{(\nu)}, \quad (57)$$

where the weights $w^{(\nu)}$ are, of course, just the probabilities for finding the marker in the respective $|\alpha_{\nu}\rangle$ states,

$$w^{(\nu)} = \langle \alpha_{\nu} | \rho_M | \alpha_{\nu} \rangle. \quad (58)$$

The identities

$$w^{(\nu)} = e^{-i\alpha_{\nu}} \langle \alpha_{\nu} | \tilde{\rho}_M | \alpha_{\nu} \rangle = e^{i\alpha_{\nu}} \langle \alpha_{\nu} | \tilde{\rho}_M^{\dagger} | \alpha_{\nu} \rangle \quad (59)$$

are the essential ingredients in showing that the right-hand sides of (47) and (57) are the same. Equations (56) and (57) tell us that there is (at least) one as-if-reality to ρ_Q in which each alternative exhibits an interference pattern with unit fringe visibility.

The question, What does a quantum eraser erase?, asked in the title of this section, is therefore answered as follows:

$$\begin{array}{l} \text{A quantum eraser removes the cover that} \\ \text{hides the as-if-reality of alternatives with} \\ \text{maximal fringe visibility.} \end{array} \quad (60)$$

If there is WW information stored in the marker beforehand, then it is erased when QE is performed. In general, however, the availability of WW information is not a precondition for QE.¹⁸

9.3. Schrödinger's cat

9.3.1. Interferences between live and dead cats? No!

The affirmative “Yes, QE is still possible.” is reassuring, but it is not a claim of practical feasibility. On the contrary, the requirement of distinguishing the eigenstates $|\alpha_{\nu}\rangle$ of $U_2^{\dagger}U_1$ from each other can and will be prohibitively difficult under the typical circumstances of an experiment. In particular, if the marker is itself a macroscopic piece of the WW detection device so that $\rho_M^{(0)}$ as well as U_1 and U_2 make reference to very many degrees of freedom, QE is simply impossible, and the interference fringes cannot be retrieved.

Such is the situation in E. Schrödinger's (in)famous cat example [8], where a radioactive atom is the quanton ($|S_1\rangle$: excited atom, $|S_2\rangle$: ground-state atom) that gets entangled with a Geiger counter, a hammer, some poisonous gas, . . . , and finally the cat. The marker consists of all this equipment plus those parts of the environment with which the Geiger counter, . . . , the cat are interacting during the period of interest.¹⁹ It is clear that any attempt to do QE on a macroscopic marker like this one,

with the aim of finding interferences between $|S_1\rangle$ and $|S_2\rangle$ is bound to fail, simply because U_1 and U_2 cannot be known with the necessary precision, if for no other reason.

Before turning to E. Schrödinger's cat problem, let us briefly mention a technical point. In general, the unitary evolution that entangles the atom (\equiv quanton) with the marker (\equiv cat plus . . .) is not of the simple form assumed in section 9.1. Rather, we begin with $\rho_Q^{(0)} = \sigma^{\dagger}\sigma = |S_1\rangle\langle S_1|$ (atom excited) and an initial marker state $\rho_M^{(0)}$ that represents our very limited knowledge about the Geiger counter, . . . , the cat plus the environment. Then

$$U_{M\&Q} = \sigma^{\dagger}\sigma V_1 + \sigma^{\dagger}V_2 + \sigma V_3 + \sigma\sigma^{\dagger}V_4 \quad (61)$$

is acting where V_1, \dots, V_4 affect only the marker variables. This results in a final P of the form (44) with

$$\begin{array}{ll} \rho_M^{(1)} \propto V_1^{\dagger}\rho_M^{(0)}V_1 & \text{(live cat),} \\ \rho_M^{(2)} \propto V_2^{\dagger}\rho_M^{(0)}V_2 & \text{(dead cat),} \\ \tilde{\rho}_M \propto V_2^{\dagger}\rho_M^{(0)}V_1 & \text{("live\&dead" cat).} \end{array} \quad (62)$$

Now, whereas $U_{M\&Q}$ is unitary, the operators V_1, \dots, V_4 need not be unitary themselves and, as a rule, they will not be. Then it is possible, and indeed plausible, that the cross term $\tilde{\rho}_M$ vanishes,²⁰ although both $\rho_M^{(1)}$ and $\rho_M^{(2)}$ are nonzero, and QE cannot be done in the first place. An elementary example for this situation (with no relevance for E. Schrödinger's cat!) is provided by a marker that has only two possible states $|M_1\rangle$ and $|M_2\rangle$; then

$$V_1 = V_4 = |M_2\rangle\langle M_1|, \quad V_2 = V_3 = |M_1\rangle\langle M_2| \quad (63)$$

are such that $U_{M\&Q}$ is unitary, and $\tilde{\rho}_M = 0$ obtains for $\rho_M^{(0)}$ of (52a) while $\rho_M^{(1)}$ and $\rho_M^{(2)}$ project to $|M_1\rangle$ and $|M_2\rangle$, respectively.

In a popular jargon any entangled state of a quantum degree of freedom and a macroscopic marker is called a Schrödinger cat.²¹ In this sense (44) represents a cat if the experimenter can distinguish between $\rho_M^{(1)}$ and $\rho_M^{(2)}$ without further ado. It thus seems that “Schrödinger cat” is just another word for an entangled system. And so one should think that everything worth saying has been said about his cat, in the more than sixty years since E. Schrödinger published his *Generalbeichte* (general confession).

²⁰In the jargon of reference [23] one could say: The entangled system suffers from *decoherence*.

²¹Sometimes the term is also applied to superpositions of macroscopically different states of a single quantum degree of freedom; see, for example, reference [40]. This usage is misleading and should be discouraged, the more so because such superpositions are common in standard interferometric devices.

¹⁸Therefore, one could lament that *quantum erasure* is a misnomer. But, who knows a more fitting term?

¹⁹That includes the quantized radiation field with its infinity of photon degrees of freedom.

Nevertheless, there is continuing interest in the subject and the question

Why do we never see interferences between the dead and the live cat? (64a)

is still being asked, which repeats (1), in essence.

Let us answer this question at the example of the Schrödinger cat (44). A look at ρ_M of (46), which is simply half the sum of $\rho_M^{(1)}$ (live cat) and $\rho_M^{(2)}$ (dead cat), justifies this immediate reply:

Because there are no interference terms in the final state of the cat. (64b)

It is true that there are such terms in P of (44), viz. the contributions involving $\tilde{\rho}_M$ (provided that $\tilde{\rho}_M$ does not vanish to begin with). But if $\rho_M^{(1)}$ and $\rho_M^{(2)}$ are macroscopically different, then the visibility \mathcal{V} is surely zero, and the statistical operator ρ_Q of (47) contains no trace of these interference terms either.

In other words: Neither a measurement on the marker (the cat) alone, nor a measurement on the quanton alone is sensitive to the presence of the interference terms in P. One would have to measure a joint observable (which one?), or better: form subensembles of cats according to the outcomes of suitable measurements on the quanton. Consider, for example, the quanton observable $Q = \sigma + \sigma^\dagger$ with eigenvalues $Q' = \pm 1$. This yields subensembles characterized by

$$\rho_M^{(\pm)} = \frac{1}{2}(\rho_M^{(1)} + \rho_M^{(2)}) \pm \frac{1}{2}(\tilde{\rho}_M^\dagger + \tilde{\rho}_M). \quad (65)$$

[Since $\rho_M^{(1)}$ and $\rho_M^{(2)}$ are assumed to be macroscopically different, $\tilde{\rho}_M$ must be traceless; see section 10. Thus, $\rho_M^{(\pm)}$ is properly normalized to unit trace.] Cross terms are present here, and so one could expect that each subensemble would exhibit interferences between $\rho_M^{(1)}$ and $\rho_M^{(2)}$ — between the live state and the dead state of each cat plus . . . , so to say.

Now, just like the cross terms in ρ_Q of (47) are noticeable only if an appropriate observable is measured, such as $O(\phi)$ of (31), the demonstration of these life/death interferences requires a corresponding marker observable. Naming this observable is easy if the marker is as simple as in sections 2–8, but utterly impossible for a macroscopic device that deserves to be called a Schrödinger cat. Phrased in words that summarize to some extent the findings of G. Süßmann [9] and A. Peres [51]: The cross terms $\pm(\tilde{\rho}_M^\dagger + \tilde{\rho}_M)$ in (65) are ineffective, they are of no phenomenological consequences; the phenomenology associated with $\rho_M^{(\pm)}$ is indiscernible from the one that goes with $\rho_M = \frac{1}{2}(\rho_M^{(+)} + \rho_M^{(-)})$ of (46).

9.3.2. Selfadjoint operators and physical observables

“Naming this observable” is not only a matter of identifying a selfadjoint operator that is sensitive to the cross

terms, which task is not so difficult, but rather a matter of finding an *observable*, that is: a physical quantity that can be measured. Whereas we take for granted that there is a corresponding selfadjoint operator to each observable²² (at least if we pay the price of some idealizations), there is no reason why we should have an observable to each selfadjoint operator.

Here I am disagreeing with P. A. M. Dirac because I think that he is asking for too much in his well-known statement (page 37 in reference [3], wording adapted to the present conventions²³):

The question now presents itself — Can every selfadjoint operator be measured? The answer theoretically is yes. In practice it may be awkward, or perhaps even beyond the ingenuity of the experimenter, to devise an apparatus which could measure some particular selfadjoint operator, but the theory always allows one to imagine that the measurement can be made.²⁴

The logical development of quantum mechanics does not need the axiom of a one-to-one correspondence between observables and selfadjoint operators. And, isn't it much more plausible, in view of the very few fundamental interactions in physics, that only a small subset of all thinkable selfadjoint operators correspond to physical observables?

²²This excuses sloppy formulations such as “the quanton observable Q” in the stead of something more precise such as “the selfadjoint operator Q that corresponds to the x component of the quanton's spin vector” and the like.

²³P. A. M. Dirac's terminology is different from the modern one; in particular, his ‘real dynamical variable’ is today's ‘Hermitian operator’ and his ‘observables’ are today's ‘selfadjoint operators.’

²⁴This succinct quote is taken from the fourth edition (1958) of P. A. M. Dirac's seminal textbook; the corresponding section of the third edition (1947) is worded identically. The second edition (1935) contains statements that amount to the same (see pages 28–30, 37 and 38), but there is nothing analogous in the first edition (1930).

It is also remarkable that P. A. M. Dirac knew about the difference between Hermitian and selfadjoint operators and that he appreciated its physical significance at the time when he completed the second edition (November 1934), but not when he wrote the first edition. Presumably, this is evidence for lessons learned from J. von Neumann's book of 1932 [6].

A different attitude can also be encountered. When the mathematical physicist K. O. Friedrichs visited W. Heisenberg in the early 1930s and told him (proudly, I imagine) that the mathematicians had made an important contribution to the development of quantum mechanics by clarifying the said difference, W. Heisenberg responded with the question: “Is there one?” (I owe this charming anecdote to R. Haag.)

A much referred-to statement to the same extent can be found in J. von Neumann’s book (page 167 of the German edition of reference [6], page 313 in the English translation):

Den physikalischen Größen eines quantenmechanischen Systems sind, wie wir wissen, die hypermaximalen Hermiteschen Operatoren eindeutig zugeordnet [...], und es ist zweckmäßig anzunehmen, daß diese Zuordnung eine ein-eindeutige ist — d. h. daß wirklich jeder hypermaximale Hermitesche Operator einer physikalischen Größe entspricht. (In III.3. machten wir hiervon gelegentlich auch Gebrauch.)²⁵

The remark in parentheses sounds as if the one-to-one correspondence — which, as I said above, is not needed as a building block of quantum mechanics — were used by J. von Neumann in an important argument. A look at his section III.3, however, reveals that this is not the case. What he has actually made use of in this section is a much weaker property, namely that if X and Y are the selfadjoint operators of two physical quantities *that can be measured simultaneously*, then there are also observables that correspond to the linear combinations $xX+yY$ with numerical coefficients x and y . Indeed, if you can simultaneously measure X and Y then you have already measured all such linear combinations. Surely, this very special case of simultaneously observable physical quantities cannot be regarded as evidence in support of the general claim of a one-to-one correspondence.

Harkening back to what is said about the Heisenberg cut in the Introduction, we note a similar (and related²⁶) situation here. No formal criterion is at hand that would enable us to judge whether any given selfadjoint operator corresponds to a physical observable.²⁷ And for reasons analogous to the ones that deny a rigorously definable location for the Heisenberg cut, such a criterion cannot exist.

²⁵In R. T. Beyer’s translation:

There corresponds to each physical quantity of a quantum mechanical system, a unique hypermaximal Hermitian operator, as we know [...], and it is convenient to assume that this correspondence is one-to-one — that is, that actually each hypermaximal operator corresponds to a physical quantity. (We also made occasional use of this in III.3.)

A “hypermaximal Hermitian” operator is a selfadjoint one.

²⁶Recall footnote 6.

²⁷Of course, a physical observable must not be in conflict with the conservation of electric charge — in other words, it must be gauge invariant — but this condition, plus a couple of similarly elementary ones, is not enough.

9.3.3. Additional remarks

In addition to this notorious problem of identifying an appropriate physical quantity of the marker to be measured, in correlation with the outcome of the Q measurement on the quanton that identifies the subensembles of (65), there is the challenge to reproduce the initial conditions implicit in $\rho_M^{(0)}$ with the precision that is necessary to avoid a complete washing-out of the interference pattern looked for. In conclusion, one must agree with E. Schrödinger’s judgment that the notion of a superposition state of a live and a dead cat is *burlesk* (ludicrous).

A remark on the notion of “superposition of two statistical operators,” such as $\rho_M^{(1)}$ and $\rho_M^{(2)}$, is in order. Superpositions of two (normalized) state vectors, $|\psi_1\rangle$ and $|\psi_2\rangle$, are familiar textbook matter: Linear combinations $|\psi\rangle = |\psi_1\rangle\alpha_1 + |\psi_2\rangle\alpha_2$, with complex coefficients, are also acceptable state vectors; the requirement

$$|\alpha_1|^2 + |\alpha_2|^2 + 2\text{Re}(\alpha_1^*\alpha_2\langle\psi_1|\psi_2\rangle) = 1 \quad (66a)$$

ensures proper normalization. Superpositions of two statistical operators, ρ_1 and ρ_2 , are constructed analogously. First one finds (Hilbert-Schmidt) operators A_1 and A_2 such that $\rho_1 = A_1^\dagger A_1$ and $\rho_2 = A_2^\dagger A_2$, then $A = \alpha_1 A_1 + \alpha_2 A_2$ yields the superposition $\rho = A^\dagger A$. Here the normalization is enforced by the analogous restriction

$$|\alpha_1|^2 + |\alpha_2|^2 + 2\text{Re}(\alpha_1^*\alpha_2 \text{tr}\{A_1^\dagger A_2\}) = 1 \quad (66b)$$

on α_1 and α_2 . For example, the statistical operators ρ_M of (46) and $\rho_M^{(\pm)}$ of (65) are superpositions of $\rho_M^{(1)}$ and $\rho_M^{(2)}$. If $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$ represent pure states, the state-vector superpositions yield $\rho = |\psi\rangle\langle\psi|$ which are particular statistical-operator superpositions; the latter kind is more general, however. Of course, there is no guarantee that arbitrary superpositions ρ of two physical states ρ_1 and ρ_2 are also physical, as is demonstrated by E. Schrödinger’s cat example.

10. Wave-particle duality

10.1. Distinguishability of the ways

For $0 < |C| < 1$, we have fringes of reduced visibility in (49). Is there also a limited amount of WW information available? Inasmuch as an interference pattern is a manifestation of the wave aspects of the quanton, whereas WW knowledge documents its particle aspects, we are heading for a quantitative statement about wave-particle duality. The principle of complementarity²⁸ implies that wave and particle aspects are mutually exclusive, in the

²⁸A technical definition is given in reference [34].

sense of (53), but it says nothing quantitative about the possible compromises.

We must read the WW marker to extract WW information, that is to say: we must measure a marker observable W [with (nondegenerate) eigenvalues W and eigenstates $|W\rangle$] and see what we can infer from the measurement result.

Suppose that the eigenvalue W is found. In view of (46), this happens with the probability

$$\langle W|\rho_M|W\rangle = \underbrace{\frac{1}{2}\langle W|\rho_M^{(1)}|W\rangle}_{\text{slit 1}} + \underbrace{\frac{1}{2}\langle W|\rho_M^{(2)}|W\rangle}_{\text{slit 2}}. \quad (67)$$

Unless the contribution of one of the slits vanishes, we cannot be certain about the way. But we know which way to bet on, namely on the one that contributes most to the probability $\langle W|\rho_M|W\rangle$.²⁹ After many repetitions, the betting odds are given by the “likelihood \mathcal{L}_W for guessing the way right,”

$$\mathcal{L}_W = \sum_W \text{Max} \left\{ \underbrace{\frac{1}{2}\langle W|\rho_M^{(1)}|W\rangle}_{\text{slit 1}}, \underbrace{\frac{1}{2}\langle W|\rho_M^{(2)}|W\rangle}_{\text{slit 2}} \right\}. \quad (68)$$

The value of \mathcal{L}_W depends, of course, on the observable W that we choose to measure.

Since we guess right for 50% of the quantons when betting at random, $\mathcal{L}_W \geq \frac{1}{2}$ must hold. And if we are lucky and know the way for each quanton with certainty, then $\mathcal{L}_W = 1$. It is therefore natural to quantify the acquired WW *knowledge* by the number

$$\mathcal{K}_W = 2\mathcal{L}_W - 1, \quad 0 \leq \mathcal{K}_W \leq 1, \quad (69)$$

so that

$$\begin{aligned} \mathcal{K}_W = 0 &: \quad \text{no WW knowledge,} \\ \mathcal{K}_W = 1 &: \quad \text{full WW knowledge.} \end{aligned} \quad (70)$$

As a consequence of (68), \mathcal{K}_W is given by³⁰

$$\mathcal{K}_W = \frac{1}{2} \sum_W \left| \langle W | (\rho_M^{(1)} - \rho_M^{(2)}) | W \rangle \right|. \quad (71)$$

The largest possible value of \mathcal{K}_W is the *distinguishability* \mathcal{D} of the ways,

$$\mathcal{D} \equiv \text{Max}_W \mathcal{K}_W. \quad (72)$$

Inasmuch as the distinguishability \mathcal{D} represents Nature’s information about the ways whereas the knowledge \mathcal{K}_W

is what Man can learn from measuring the observable W , the inequality

$$\mathcal{K}_W \leq \mathcal{D} \quad (73)$$

states an obvious hierarchy: Man cannot be smarter than Nature. In passing we note that the numbers \mathcal{L}_W , \mathcal{K}_W , and \mathcal{D} quantify information or knowledge without invoking an entropic concept of some kind.

The equal sign holds in (73) when the eigenstates $|W\rangle$ of W are also eigenstates of $\rho_M^{(1)} - \rho_M^{(2)}$; then the moduli of the eigenvalues of $\rho_M^{(1)} - \rho_M^{(2)}$ are summed in (71), so that the distinguishability is explicitly given by

$$\mathcal{D} = \frac{1}{2} \text{tr}_M \left\{ \left| \rho_M^{(1)} - \rho_M^{(2)} \right| \right\}. \quad (74)$$

Thus, mathematically speaking, \mathcal{D} is the distance between $\frac{1}{2}\rho_M^{(1)}$ and $\frac{1}{2}\rho_M^{(2)}$ in the trace-class norm.

10.2. A digression: Asymmetric interferometers

Which-way information of a different kind is available in asymmetric interferometers where the a priori probabilities of the alternatives “through slit 1” and “through slit 2” are different. Then the way is predictable to some extent, so that we have some WW knowledge even without any WW marking.

In case of such an asymmetry, the initial statistical operator of the quanton is of the general form

$$\rho_{Q,\text{asym}}^{(0)} = w_1 \sigma^\dagger \sigma + w_2 \sigma \sigma^\dagger + \sqrt{w_1 w_2} (\epsilon \sigma + \epsilon^* \sigma^\dagger), \quad (75)$$

where w_1 and w_2 are the respective probabilities for the two ways ($w_1 + w_2 = 1$, of course). The parameter ϵ plays the same role here as in equation (9), which is the $w_1 = w_2 = \frac{1}{2}$ version of (75). When betting on the more probable way, we get an a priori likelihood of

$$\mathcal{L}_{\text{a priori}} = \text{Max}\{w_1, w_2\} \equiv \frac{1}{2}(1 + \mathcal{P}), \quad (76)$$

which identifies the *predictability* \mathcal{P} of the ways,³¹

$$\mathcal{P} = |w_1 - w_2|. \quad (77)$$

With a WW detection device in place, corresponding expressions for \mathcal{K}_W and \mathcal{D} are found. They are obtained from (71) and (74), respectively by the replacements $\frac{1}{2}\rho_M^{(1)} \rightarrow w_1\rho_M^{(1)}$ and $\frac{1}{2}\rho_M^{(2)} \rightarrow w_2\rho_M^{(2)}$. The inequality (73) is then supplemented by

²⁹This betting strategy was introduced by W. K. Wootters and W. H. Zurek [52].

³⁰The identity $\text{Max}\{x, y\} = \frac{1}{2}(x + y) + \frac{1}{2}|x - y|$ is used in the transition from (68) to (71).

³¹If something is known about the process of formation of the mixture $\rho_{Q,\text{asym}}^{(0)}$, that is: if a particular blend can be physically distinguished, then the a priori likelihood and thus the predictability can be larger; see reference [38] for a discussion of such situations. Equations (78) and (80) are equally valid under these circumstances.

$$\mathcal{P} \leq \mathcal{K}_W, \quad (78)$$

which sets an obvious lower bound on the knowledge \mathcal{K}_W .

For the statistical operator (75) the fringes have an a priori visibility \mathcal{V}_0 that is given by

$$\mathcal{V}_0 = 2\sqrt{w_1 w_2} |\epsilon|. \quad (79)$$

As a consequence of the positivity of $\rho_{Q,\text{asym}}^{(0)}$ — which requires $|\epsilon| \leq 1$ — the predictability \mathcal{P} and the a priori visibility \mathcal{V}_0 must obey the inequality

$$\mathcal{P}^2 + \mathcal{V}_0^2 \leq 1. \quad (80)$$

This observation has been made — implicitly or explicitly — by a number of authors in various physical contexts in which alternatives can become predictable to some extent. I am aware of references [38,52–59]; the measurements on neutrons and photons reported in [53,54] and [57], respectively, are consistent with (80).

10.3. An inequality

The fringe visibility \mathcal{V} of (51) and the distinguishability \mathcal{D} of (74) quantify the wave aspects of the quanton and its particle aspects, respectively, and so the stage is set for the quantitative statement about wave-particle duality. It reads [60]

$$\mathcal{D}^2 + \mathcal{V}^2 \leq 1. \quad (81)$$

Clearly, the extreme cases of (53), *viz*

$$\begin{aligned} \mathcal{V} = 1 & \quad \text{implies} \quad \mathcal{D} = 0, \\ \mathcal{D} = 1 & \quad \text{implies} \quad \mathcal{V} = 0, \end{aligned} \quad (82)$$

are an immediate consequence of (81), and there is room for the example of equations (52) in which $\mathcal{V} = 0$ and $\mathcal{D} = 0$.

The equal sign holds in (81) if $\rho_M^{(0)}$ represents a pure state, because then we have³²

$$\begin{aligned} \rho_M^{(0)} &= |M^{(0)}\rangle\langle M^{(0)}|, \\ \rho_M^{(1)} &= |M^{(1)}\rangle\langle M^{(1)}| \quad \text{with} \quad |M^{(1)}\rangle = U_1^\dagger |M^{(0)}\rangle, \\ \rho_M^{(2)} &= |M^{(2)}\rangle\langle M^{(2)}| \quad \text{with} \quad |M^{(2)}\rangle = U_2^\dagger |M^{(0)}\rangle, \\ \tilde{\rho}_M &= |M^{(2)}\rangle\langle M^{(1)}|, \end{aligned} \quad (83a)$$

and the non-zero eigenvalues of $\rho_M^{(1)} - \rho_M^{(2)}$ are given by $\pm(1 - |\langle M^{(1)}|M^{(2)}\rangle|)^{1/2}$, so that

³²Note that $|M^{(1)}\rangle$ and $|M^{(2)}\rangle$ need not be orthogonal to each other. An analogous remark applies to $|M_k^{(1)}\rangle$ and $|M_k^{(2)}\rangle$ in (86).

$$\mathcal{D} = \sqrt{1 - |\langle M^{(1)}|M^{(2)}\rangle|^2} \quad (83b)$$

and

$$\mathcal{V} = |\text{tr}_M \{\tilde{\rho}_M\}| = |\langle M^{(1)}|M^{(2)}\rangle|. \quad (83c)$$

Indeed, the upper limit of (81) is reached.

For a proof of the duality relation (81) we follow the strategy of [60]; situations that are more general than the ones considered here — in particular, the extension to asymmetric interferometers and the complications arising from quanton-marker couplings of the form (61) — are dealt with in references [61,62]. First, we employ the spectral decomposition of $\rho_M^{(0)}$,

$$\rho_M^{(0)} = \sum_k m_k |M_k\rangle\langle M_k| \quad (84a)$$

with

$$m_k \geq 0, \quad \sum_k m_k = 1, \quad \langle M_j|M_k\rangle = \delta_{jk}, \quad (84b)$$

in $\rho_M^{(1)}$ and $\rho_M^{(2)}$ to arrive at

$$\rho_M^{(1)} - \rho_M^{(2)} = \sum_k m_k \left(|M_k^{(1)}\rangle\langle M_k^{(1)}| - |M_k^{(2)}\rangle\langle M_k^{(2)}| \right) \quad (85)$$

where

$$|M_k^{(1)}\rangle = U_1^\dagger |M_k^{(0)}\rangle, \quad |M_k^{(2)}\rangle = U_2^\dagger |M_k^{(0)}\rangle. \quad (86)$$

Then we make use of the triangle inequality

$$\text{tr} \{ |\rho_a - \rho_b| \} \leq \text{tr} \{ |\rho_a| \} + \text{tr} \{ |\rho_b| \}, \quad (87)$$

valid for any two trace-class operators ρ_a and ρ_b , to establish

$$\mathcal{D} \leq \frac{1}{2} \sum_k m_k \text{tr}_M \left\{ \left| |M_k^{(1)}\rangle\langle M_k^{(1)}| - |M_k^{(2)}\rangle\langle M_k^{(2)}| \right| \right\}. \quad (88)$$

The lesson learned at the example of equations (83) enables us to evaluate these simpler traces, and we find

$$\mathcal{D} \leq \sum_k m_k \sqrt{1 - |\langle M_k^{(1)}|M_k^{(2)}\rangle|^2}. \quad (89)$$

It is convenient to express the amplitudes $\langle M_k^{(1)}|M_k^{(2)}\rangle$ in terms of two angle variables ϑ_k and φ_k ,

$$\langle M_k^{(1)}|M_k^{(2)}\rangle = \sin \vartheta_k e^{i\varphi_k} \quad (90)$$

with $0 \leq \vartheta_k \leq \pi/2$ and $0 \leq \varphi_k < 2\pi$, so that

$$\mathcal{D} \leq \sum_k m_k \cos \vartheta_k. \quad (91)$$

Likewise, the visibility is given by

$$\mathcal{V} = \left| \sum_k m_k \langle M_k^{(1)} | M_k^{(2)} \rangle \right| = \left| \sum_k m_k \sin \vartheta_k e^{i\varphi_k} \right|. \quad (92)$$

In conjunction with (91) this yields

$$\mathcal{D}^2 + \mathcal{V}^2 \leq \sum_j m_j \sum_k m_k [\dots], \quad (93)$$

where $[\dots] = \cos \vartheta_j \cos \vartheta_k + \sin \vartheta_j \sin \vartheta_k \cos(\varphi_j - \varphi_k)$ is the scalar product of two unit vectors in spherical coordinates, so that $[\dots] \leq 1$, and we get

$$\mathcal{D}^2 + \mathcal{V}^2 \leq \left(\sum_k m_k \right)^2 = \left(\text{tr}_M \left\{ \rho_M^{(0)} \right\} \right)^2 = 1, \quad (94)$$

which is (81), indeed.

We emphasize that the proof of the duality relation (81) does not invoke an uncertainty relation of the Heisenberg-Robertson kind [63,64], that is:

$$\delta X \delta Y \geq \frac{1}{2} |\langle i[X, Y] \rangle| \quad (95)$$

for the spreads of two observables X and Y and the expectation value of their commutator, and the same remark applies to the more general treatments in [61,62]. Indeed, the mathematics used in demonstrating (95) is quite different and more elementary: One notes that the expectation value $\langle A^\dagger A \rangle$ is nonnegative and exploits this fact for $A = \delta Y(X - \langle X \rangle) \pm i\delta X(Y - \langle Y \rangle)$. The conclusion

The duality relation (81) and the uncertainty relation (95) are logically independent statements. Both are consequences of the rules of the game we call quantum mechanics, but one does not imply the other. (96)

is justified by this observation.

One should appreciate that (81) leaves a lot of room for simultaneous manifestations of wave and particle properties between the two extreme cases of (53) or (82). For example, we could have $\mathcal{K}_W = \mathcal{D} = 80\%$ so that we can guess the way right for $(1 + \mathcal{D})/2 = 90\%$ of the quantons while building up an interference pattern with a fringe visibility of as much as $\mathcal{V} = \sqrt{1 - \mathcal{D}^2} = 60\%$. And even if we know the way with a confidence of $(1 + \mathcal{D})/2 = 99\%$, we may still have well visible fringes with $\mathcal{V} = 20\%$. In the related, yet somewhat different, context of (80) D. M. Greenberger and A. Yasin [58] remark that this “amazing result testifies to the power of the superposition principle” — a good line to end on.

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Dedication

It is a great pleasure indeed to dedicate these notes to Professor Georg Süßmann on the occasion of his 70th birthday. His powerful performance in discussions, always having an enlightening example or analogy at his disposal, can shake anybody’s self-confidence — but then one is amply rewarded by emerging with convictions that are battle tested.

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