1 We need to eliminate the fugacity \( z \) from

\[
\rho \lambda^3 = g_2^z(z) = z + 2^{-3/2}z^2 + \cdots \quad \text{and} \quad \beta P \lambda^3 = g_2^z(z) = z + 2^{-5/2}z^2 + \cdots
\]

for \( 0 < z \ll 1 \). This gives first \( z \approx \rho \lambda^3 - 2^{-3/2}(\rho \lambda^3)^2 \), and then \( \beta P \lambda^3 \approx \rho \lambda^3 - 2^{-3/2}(\rho \lambda^3)^2 + 2^{-5/2}(\rho \lambda^3)^2 \) or, finally,

\[
\beta P \approx \rho - 2^{-5/2}\rho^2 \lambda^3.
\]

2 The canonical partition function is

\[
Q(\beta, V, N) = \frac{1}{N!} \left[ \frac{V}{(2\pi \hbar)^3} \int (d\mathbf{p}) e^{-\beta c(|\mathbf{p}|)} \right]^N = \frac{1}{N!} \left[ \frac{V}{(2\pi \hbar)^3} (\beta c)^3 \right]^N,
\]

and the free energy is

\[
F(\beta, V, N) = -\frac{1}{\beta} \log Q = - \frac{N}{\beta} + \frac{N}{\beta} \log \frac{\pi^2 (\beta \hbar c)^3}{V/N}.
\]

This yields

\[
P = -\frac{\partial F}{\partial V} = \frac{N}{\beta V} \quad \text{and} \quad U = F + TS = F + \beta \frac{\partial F}{\partial \beta} = \frac{\partial (\beta F)}{\partial \beta} = \frac{3N}{\beta},
\]

so that \( u = \frac{U}{V} = 3P \). This is as expected, since we have a state density \( \propto p^2 dp \propto \varepsilon^2 d\varepsilon \) with \( \kappa = 3 \) in (3.9.3) and (3.9.6).

3 We consider \( E_{TF}[\rho] - E_{TF}[\rho_{TF}] = \left( E_{TF}[\rho_{TF} + x(\rho - \rho_{TF})] - E_{TF}[\rho_{TF}] \right)_{x=1} \equiv f(x)_{x=1} \), and note that \( f(1) = f(0) + f'(0) + \frac{1}{2} f''(y) \) with \( 0 \leq y \leq 1 \). Here, \( f(0) = 0 \) by construction and \( f'(0) = 0 \) since \( E_{TF}[\rho] \) is stationary at \( \rho_{TF} \), and we need to verify that \( f''(y) \geq 0 \). With \( \Delta(r) = \rho(r) - \rho_{TF}(r) \), we have

\[
f''(y) = \frac{\hbar^2}{10\pi^2 m} \int (d\mathbf{r}) (3\pi^2)^{5/3} \frac{5}{3} \frac{2}{3} \Delta(r)^2 [\rho_{TF}(r) + y\Delta(r)]^{-1/3}
\]

\[
+ \frac{\varepsilon^2}{2} \int (d\mathbf{r})(d\mathbf{r}') \frac{\Delta(r) \Delta(r')}{|\mathbf{r} - \mathbf{r}'|},
\]

where both terms are positive. It follows that \( f''(y) \geq 0 \).
(a) Proceeding from

\[ E_k = \sum_j \begin{cases} 
+J & \text{if } s_j s_{j+1} = -1 \\
-J_+ & \text{if } s_j = s_{j+1} = +1 \\
-J_- & \text{if } s_j = s_{j+1} = -1 
\end{cases} \]

\[ = \frac{1}{4} \sum_j \left[ (2J - J_+ - J_-) + (J_- - J_+) (s_j + s_{j+1}) - (2J + J_+ + J_-) s_j s_{j+1} \right] \]

we identify

\[ \mathcal{E} = \frac{1}{4} (2J - J_+ - J_-), \quad E_0' = J_- - J_+, \quad J' = \frac{1}{4} (2J + J_+ + J_-). \]

(b) The partition function is

\[ Q(\beta J, \beta J_+, \beta J_-, N) = e^{-N \mathcal{E} \lambda_+ (\beta E_0', \beta J')} \]

with \( \lambda_+ (\beta E_0, \beta J) \) from (4.2.36) in (4.2.38), where the \( \lambda_-^N \) term is negligibly small.

(c) Here we have \( \mathcal{E} = 0, E_0' = -2\epsilon \), and \( J' = J \). In \( F = N \mathcal{E} - \frac{N}{\beta} \log \lambda_+ (\beta E_0', \beta J') \),

we need \( \lambda_+ (\beta E_0, \beta J) \) to first-order in \( \beta E_0 \), which is \( \lambda_+ (0, \beta J) = 2 \cosh(\beta J) \) as there are no first-order terms. Accordingly, we obtain

\[ F = -\frac{N}{\beta} \log \left( 2 \cosh(\beta J) \right) + \cdots , \]

where the ellipsis stands for terms of second and higher order in \( \epsilon \).