Problem 1 (25=5+15+5 marks)
For a certain rubber band of length \( L \) that consists of \( n \) moles of rubber, the differential of the internal energy \( U \) is
\[
dU = T\,dS + \tau\,dL + \mu\,dn,
\]
where \( T, S, \tau, \mu \) are the temperature, entropy, tension, and chemical potential, respectively.

(a) Show that
\[
U(S, L, n) = T(S, L, n)S + \tau(S, L, n)L + \mu(S, L, n)n.
\]

(b) The equation of state \( TS = 3\tau L \) holds for this rubber band. It is also known that \( \tau L^{1/2} \) does not change if both \( S \) and \( n \) are kept constant (no heat transfer, no particle exchange). Find \( U(S, L, n) \) up to a multiplicative constant.

(c) Confirm that the equation in part (a) holds for this \( U(S, L, n) \).

Problem 2 (25=15+10 marks)
The so-called Berthelot gas has the equation of state
\[
p = \frac{RT}{v - b} - \frac{a}{v^2 RT} \quad \text{with} \quad a, b > 0,
\]
where, as usual, \( p, T, v, \) and \( R \) are pressure, temperature, molar volume, and gas constant, respectively. Above a critical temperature \( T_{cr} \), this gives a pressure that decreases monotonically as the molar volume increases. For temperatures \( T \) below \( T_{cr} \), the Maxwell construction identifies a range \( v^{(1)} \leq v \leq v^{(2)} \), in which the pressure does not depend on \( v \). For given material constants \( a \) and \( b \), this co-existence pressure \( p(T) \) is a function of temperature \( T \). At the critical temperature, we have the critical pressure, \( p(T_{cr}) = p_{cr} \), and the corresponding critical molar volume \( v_{cr} \).

(a) Find \( T_{cr}, p_{cr}, \) and \( v_{cr} \) in terms of \( a \) and \( b \). What is the value of \( p_{cr}v_{cr}/T_{cr} \)?

(b) For temperatures \( T \) just below the critical temperature, \( T_{cr} - T \ll T_{cr} \), the co-existence pressure is well approximated by
\[
p(T) = p_{cr} \left( \frac{xT}{T_{cr}} - (x - 1) \right).
\]
Determine the number \( x \).
Problem 3 (25=15+10 marks)
Note: A homework exercise dealt with the link between the partition functions of the canonical and grand canonical ensembles. This problem concerns the link between the microcanonical and the canonical ensembles.

A thermodynamical system is described by entropy $S$ or temperature $T = 1/(k_B \beta)$ together with other extensive variables $X$. As usual, we denote the count of microstates with energy $E$ by $\Omega(E, X)$ and the canonical partition function by $Q(\beta, X)$.

(a) Show that, for given $Q(\beta, X)$,

$$\Omega(E, X) = Q(\beta, X)e^{\beta E} \quad \text{with } \beta \text{ such that } EQ(\beta, X) = -\frac{\partial Q(\beta, X)}{\partial \beta}.$$ 

(b) What, in turn, is $Q(\beta, X)$ for known $\Omega(E, X)$?

Problem 4 (25=15+10 marks)
For ideal gases, we know from lecture that the average occupation number is

$$\langle n_j \rangle = \frac{1}{e^{\beta (\varepsilon_j - \mu)} \pm 1} \begin{cases} 
\text{for fermions} \\
\text{for bosons}
\end{cases},$$

where $\varepsilon_j$ is the $j$th single-particle energy and $\mu$ is the chemical potential. In the boson case, assume that the temperature is sufficiently high that there is no Bose-Einstein condensation.

(a) What is the corresponding expression for the correlation $\langle n_j n_{j'} \rangle - \langle n_j \rangle \langle n_{j'} \rangle$?

(b) Find the corresponding fluctuation $\langle \delta N^2 \rangle$ in the total number of particles, and verify that it is consistent with the general expression (page 85 in the lecture notes)

$$\langle \delta N^2 \rangle = \left( \frac{\partial \langle N \rangle}{\partial (\beta \mu)} \right)_{V, \beta}.$$