(a) Since $U, S, L, n$ are extensive variables, we have first

$$U(\lambda S, \lambda L, \lambda n) = \lambda U(S, L, n) \quad \text{for} \quad \lambda > 0$$

and then

$$U(S, L, n) = \frac{\partial}{\partial \lambda} U(\lambda S, \lambda L, \lambda n) \bigg|_{\lambda=1}$$

$$= S \left( \frac{\partial U}{\partial S} \right)_{L,n} + L \left( \frac{\partial U}{\partial L} \right)_{S,n} + n \left( \frac{\partial U}{\partial n} \right)_{S,L}$$

$$= ST + L \tau + n \mu,$$

where $T, \tau, \mu$ are functions of $S, L, n$.

(b) In terms of $U(S, L, n)$, the equation of state is the differential equation

$$\left( S \frac{\partial}{\partial S} - 3L \frac{\partial}{\partial L} \right) U(S, L, n) = 0,$$

which is solved by any function of $S^3L$. Proper scaling requires the form

$$U(S, L, n) = nf(S^3L/n^4)$$

with an undetermined function $f(\ )$. We also know that

$$\tau L^{1/2} = L^{1/2} \frac{\partial}{\partial L} U(S, L, n) = \frac{1}{2} \frac{\partial}{\partial \sqrt{L}} U(S, L, n)$$

does not depend on $L$, which tells us that

$$f(S^3L/n^4) = (\text{const}) \sqrt{S^3L/n^4} + (\text{const'}) .$$

We can put $(\text{const'}) = 0$ because this contribution to

$$U(S, L, n) = (\text{const}) \sqrt{S^3L/n^2} + (\text{const'}) n$$

is of no thermodynamic consequence — it is just a fixed energy per rubber particle.
For \( U \propto \sqrt{S^3L/n^2} \), we have
\[
TS = S \frac{\partial U}{\partial S} = \frac{3}{2} U, \quad \tau L = L \frac{\partial U}{\partial L} = \frac{1}{2} U, \quad \mu n = n \frac{\partial U}{\partial n} = -U
\]
and
\[
TS + \tau L + \mu n = \left( \frac{3}{2} + \frac{1}{2} - 1 \right) U = U.
\]

(a) For the critical values, the first and the second derivative of the right-hand side of the equation of state with respect to \( v \) must equal 0. Except for the replacement \( a \to a/(RT_{cr}) \), the pair of equations is the same as for the van der Waals gas. Therefore, we have
\[
v_{cr} = 3b \quad \text{and} \quad RT_{cr} = \frac{8a}{27bRT_{cr}},
\]
so that
\[
RT_{cr} = \sqrt{\frac{8a}{27b}} = \frac{2}{3} \sqrt{\frac{2a}{3b}} \quad \text{and} \quad p_{cr} = \frac{1}{12b} \sqrt{\frac{2a}{3b}}.
\]
It follows that
\[
\frac{p_{cr}v_{cr}}{T_{cr}} = \frac{3}{8} R.
\]

(b) With \( p(T, v) \) given by the equation of state, we have
\[
p(T) = p(T, v^{(1)}(T)) = p(T, v^{(2)}(T)).
\]
Accordingly,
\[
\frac{xp_{cr}}{T_{cr}} = \left. \frac{\partial p(T)}{\partial T} \right|_{T=T_{cr}} = \left( \frac{\partial p}{\partial T} \right)_{T_{cr}, v_{cr}} (T_{cr}, v_{cr}) + \left( \frac{\partial p}{\partial v} \right)_{T_{cr}, v_{cr}} \left. \frac{dv^{(1 or 2)}}{dT} \right|_{T=T_{cr}},
\]
where the second term vanishes at the critical point, and the first term gives
\[
\frac{xp_{cr}}{T_{cr}} = \frac{R}{v_{cr} - b} + \frac{aR}{(v_{cr}RT_{cr})^2} = \frac{R}{2b} + \frac{3R}{8b} = \frac{7R}{8b}.
\]
Since \( p_{cr}/T_{cr} = R/(8b) \), we have \( x = 7 \) and
\[
\bar{p}(T) = p_{cr} \left( 7 \frac{T}{T_{cr}} - 6 \right) \quad \text{for} \quad T_{cr} - T \ll T_{cr}.
\]
For simplicity, we keep the common $X$ dependence implicit, that is: $\Omega(E)$ stands for $\Omega(E, X)$, $Q(\beta)$ stands for $Q(\beta, X)$, etc.

(a) We have $\beta F(\beta) = -\log Q(\beta)$, $S = -\frac{\partial F}{\partial T}$, and $(U =) E = F + TS$. Therefore,

$$\frac{S}{k_B} = \log \Omega(E) = \beta E - \beta F = \beta E + \log Q(\beta) \quad \text{or} \quad \Omega(E) = Q(\beta) e^{\beta E}$$

with

$$E = F - T \frac{\partial F}{\partial T} = E + \beta \frac{\partial F}{\partial \beta} = \frac{\partial (\beta F)}{\partial \beta} = - \frac{1}{Q(\beta)} \frac{\partial Q(\beta)}{\partial \beta}$$

or

$$EQ(\beta) = - \frac{\partial Q(\beta)}{\partial \beta}.$$

(b) Now we have $S = k_B \log \Omega(E)$, $\beta = \frac{\partial (S/k_B)}{\partial E} = \frac{\partial \log \Omega(E)}{\partial E}$, and

$$\log Q(\beta) = -\beta F(\beta) = \beta (TS - E) = \log \Omega(E) - \beta E$$

or $Q(\beta) = \Omega(E) e^{-\beta E}$

with $E$ such that

$$\beta = \frac{1}{\Omega(E)} \frac{\partial \Omega(E)}{\partial E} \quad \text{or} \quad \beta \Omega(E) = \frac{\partial \Omega(E)}{\partial E}.$$

Since

$$\langle n_j \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \varepsilon_j} \log Z = -\frac{1}{\beta Z} \frac{\partial Z}{\partial \varepsilon_j},$$

we have

$$\langle n_j n_{j'} \rangle = \frac{1}{\beta^2 Z} \frac{\partial}{\partial \varepsilon_j} \frac{\partial}{\partial \varepsilon_{j'}} (\langle n_{j'} \rangle Z) = -\frac{1}{\beta Z} \frac{\partial \langle n_{j'} \rangle}{\partial \varepsilon_j} \langle n_j \rangle + \langle n_j \rangle \langle n_{j'} \rangle,$$

so that

$$\langle n_j n_{j'} \rangle - \langle n_j \rangle \langle n_{j'} \rangle = -\frac{1}{\beta} \frac{\partial \langle n_{j'} \rangle}{\partial \varepsilon_j} = -\delta_{jj'} \frac{1}{\beta} \frac{\partial \langle n_j \rangle}{\partial \varepsilon_j} = \delta_{jj'} \langle n_j \rangle^2 \frac{\partial (\langle n_j \rangle)^{-1}}{\partial \varepsilon_j}$$

$$= \delta_{jj'} \langle n_j \rangle^2 \beta e^{\beta (\varepsilon_j - \mu)} = \delta_{jj'} \langle n_j \rangle^2 \left( \frac{1}{\langle n_j \rangle} \mp 1 \right)$$

$$= \delta_{jj'} \langle n_j \rangle \left( 1 \mp \langle n_j \rangle \right).$$
We have

\[ \langle \delta N^2 \rangle = \sum_{j,j'} \left( \langle n_j n_{j'} \rangle - \langle n_j \rangle \langle n_{j'} \rangle \right) = \sum_j \langle n_j \rangle (1 \mp \langle n_j \rangle) = \langle N \rangle \mp \sum_j \langle n_j \rangle^2, \]

and said consistency follows from

\[
\frac{\partial \langle N \rangle}{\partial (\beta \mu)} = \sum_j \frac{\partial \langle n_j \rangle}{\partial (\beta \mu)} = -\sum_j \langle n_j \rangle^2 \frac{\partial}{\partial (\beta \mu)} e^{\beta \varepsilon_j - \beta \mu} = \sum_j \langle n_j \rangle^2 e^{\beta \varepsilon_j - \beta \mu} = \sum_j \langle n_j \rangle^2 \left( \langle n_j \rangle^{-1} + 1 \right) = \sum_j \langle n_j \rangle \left( 1 \mp \langle n_j \rangle \right). \]