(a) On the way up \((0 \leq t \leq T)\), the speed \(v(t)\) changes in accordance with
\[
\dot{v} = -g - g \frac{v^2}{v_\infty^2}
\]
or
\[
\dot{v}/v_\infty = -\frac{g}{v_\infty}
\]
so that
\[
\tan^{-1}\left(\frac{v(t)}{v_\infty}\right) + gt = \text{constant}
\]
or
\[
\frac{v(t)}{v_\infty} = \tan\left(\frac{g}{v_\infty} (T - t)\right) = v_\infty \frac{d}{dt} \log \left( \cos\left(\frac{g}{v_\infty} (T - t)\right)^2 \right)
\]
with \(\tan\left(\frac{gT}{v_\infty}\right) = \frac{v_0}{v_\infty}\).

It follows that the height reached is
\[
h = \int_0^T dt \ v(t) = -\frac{v_\infty^2}{2g} \log \left( \cos\left(\frac{gT}{v_\infty}\right)^2 \right) = \frac{v_\infty^2}{2g} \log \left( 1 + \tan\left(\frac{gT}{v_\infty}\right)^2 \right)
\]
or, finally,
\[
h = \frac{v_\infty^2}{2g} \log \frac{v_\infty^2 + v_0^2}{v_\infty^2}.
\]

(b) On the way down \((t \geq T)\), the speed \(v(t)\) changes in accordance with
\[
\dot{v} = g - g \frac{v^2}{v_\infty^2}
\]
or
\[
\dot{v}/v_\infty = \frac{g}{v_\infty}
\]
so that
\[
\tanh^{-1}\left(\frac{v(t)}{v_\infty}\right) - \frac{gt}{v_\infty} = \text{constant}
\]
or
\[
\frac{v(t)}{v_\infty} = \tanh\left(\frac{g}{v_\infty} (t - T)\right) = v_\infty \frac{d}{dt} \log \left( \cosh\left(\frac{g}{v_\infty} (t - T)\right)^2 \right).
\]

It follows that the height above ground at time \(t\) is
\[
h - \int_t^T dt' \ v(t') = h - \frac{v_\infty^2}{2g} \log \left( \cosh\left(\frac{g}{v_\infty} (t - T)\right)^2 \right)
\]
\[
= h + \frac{v_\infty^2}{2g} \log \left( 1 - \tanh\left(\frac{g}{v_\infty} (t - T)\right)^2 \right)
\]
\[
= h - \frac{v_\infty^2}{2g} \log \frac{v_\infty^2}{v_\infty^2 - v(t)^2}.
\]
Therefore, $v_1$ is given by
$$h = \frac{v_0^2}{2g} \log \frac{v_\infty^2}{v_\infty^2 - v_1^2}.$$  

It follows that
$$\frac{v_\infty^2 + v_0^2}{v_\infty^2} = \frac{v_\infty^2}{v_\infty^2 - v_1^2} \quad \text{or} \quad v_1 = \frac{v_0 v_\infty}{\sqrt{v_\infty^2 + v_0^2}}.$$  

2

(a) At equilibrium, the masses are distance $a$ apart and each at distance $a$ from the adjacent wall, and all springs are relaxed; we choose $x_1$ and $x_2$ as the displacements from equilibrium to the right. Then
$$L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k_1 (x_1^2 + x_2^2) + \frac{1}{2} k_2 (x_2 - x_1)^2 = \frac{1}{2} \dot{X}^T M \dot{X} - \frac{1}{2} \dot{X}^T K X$$
with
$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{pmatrix}.$$  

(b) Since $X^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $X^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are common eigencolumns of $M$ and $K$, they specify the normal modes, and we find the characteristic frequencies $\omega_1$ and $\omega_2$ from
$$0 = (\omega_1^2 M - K) X^{(1)} = (m \omega_1^2 - k_1) X^{(1)} \quad \text{so that} \quad \omega_1 = \sqrt{\frac{k_1}{m}},$$
$$0 = (\omega_2^2 M - K) X^{(2)} = (m \omega_2^2 - k_1 - 2k_2) X^{(2)} \quad \text{so that} \quad \omega_2 = \sqrt{\frac{k_1 + 2k_2}{m}}.$$  

Since $\omega_1 < \omega_2$, the first mode is the slow one, and the second mode is the fast one.

(c) The first normal mode is just center-of-mass motion, where the distance between the masses is $a$ at all times and the inner spring is always relaxed:

![Diagram of first normal mode]

The second normal mode is a breathing mode, where the center-of-mass is at rest and the two masses move with opposite velocities:

![Diagram of second normal mode]
The Hamilton function is
\[ H = \frac{1}{2} P^T M^{-1} P + \frac{1}{2} X^T K X = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{1}{2} k_1 (x_1^2 + x_2^2) + \frac{1}{2} k_2 (x_2 - x_1)^2. \]

(b) The center-of-mass of the two-body system is at
\[ R = \frac{1}{M_1 + M_2} (M_1 R_1 + M_2 R_2), \]
and the positions of bodies 1 and 2 relative to the center-of-mass are
\[ R_1 - R = \frac{M_2}{M_1 + M_2} (R_1 - R_2) \quad \text{and} \quad R_2 - R = \frac{M_1}{M_1 + M_2} (R_2 - R_1). \]

Upon applying Steiner’s theorem twice, we get
\[ I = I_1 + I_2 + \frac{M_1 M_2}{M_1 + M_2} [(R_1 - R_2)^2 - (R_1 - R_1) (R_2 - R_2)]. \]

(b) We denote the position vectors of the four point masses by \( r_1, r_2, r_3, \) and \( r_4. \) Their cartesian coordinates could be
\[ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \frac{a}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \]
for example, and we have \( r_j \cdot r_k = a^2 \delta_{jk} - \frac{1}{4} a^2 \) as well as \( \sum_{j=1}^4 r_j \cdot r_j = a^2 1 \) for their dot products and the sum of their dyadic squares. Accordingly, the inertia dyadic is
\[ I_{\text{4 masses}} = \sum_{j=1}^4 m (r_j^2 1 - r_j \cdot r_j) = 2ma^2 1. \]

(c) We recognize the situation of part (a) for the three-mass system as body 1 \((M_1 = 3m, R_1 = -\frac{1}{3} r_4, \) and \( l_1 = l_{\text{3 masses}})\) and the fourth mass as body 2 \((M_2 = m, R_2 = r_4, \) and \( l_2 = l_{\text{2 masses}})\), and the two-body system is the four-mass system. With \( a^2 = \frac{4}{3} r_4^2 \), we then have
\[ l_1 \text{masses} = \frac{8}{3} m r_2^2 \mathbf{1} = l_3 \text{masses} + 0 + \frac{3m^2}{4m} \left[ \left( \frac{4}{3} r_4 \right)^2 \mathbf{1} - \frac{4}{3} r_4 \frac{4}{3} r_4 \right] \]

and find

\[ l_3 \text{masses} = \frac{4}{3} m (r_4^2 \mathbf{1} + r_4 r_4) . \]

(a) In the laboratory frame we have

\[ m \ddot{r} = -\nabla V = -m \omega_0^2 \left[ r - 3 n \cdot r \right] . \]

(b) We introduce coordinates in the rotating frame by writing

\[ r = x n + y e_z \times n + z e_z \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \]

With

\[ \frac{d}{dt} n = \Omega e_z \times n \quad \text{and} \quad \frac{d}{dt} e_z \times n = -\Omega n \]

we then have

\[ \dot{r} \equiv \begin{pmatrix} \dot{x} - \Omega y \\ \dot{y} + \Omega x \\ \dot{z} \end{pmatrix} , \quad \ddot{r} \equiv \begin{pmatrix} \ddot{x} - 2\Omega \dot{y} - \Omega^2 x \\ \ddot{y} + 2\Omega \dot{x} - \Omega^2 y \\ \ddot{z} \end{pmatrix} \quad \text{and} \quad r - 3 n \cdot r \equiv \begin{pmatrix} -2x \\ y \\ z \end{pmatrix} . \]

Together they gives us the equation of motion

\[ \begin{pmatrix} \ddot{x} - 2\Omega \dot{y} - (2\omega_0^2 + \Omega^2)x \\ \ddot{y} + 2\Omega \dot{x} + (\omega_0^2 - \Omega^2)y \\ \ddot{z} + \omega_0^2 z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \]

(c) The \( z \) motion is harmonic all by itself and thus stable, irrespective of the value of \( \Omega \).

For the coupled \( xy \) motion, we make the exponential ansatz \( \begin{pmatrix} x \\ y \end{pmatrix} = e^{i\omega t} \begin{pmatrix} a \\ b \end{pmatrix} \),

where \( \omega \) must be real to ensure that the point mass stays near \( r = 0 \). The ansatz works if

\[ \begin{pmatrix} \Omega^2 + 2\omega_0^2 + \omega^2 & 2i\Omega \omega \\ -2i\Omega \omega & \Omega^2 - \omega_0^2 + \omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 , \]

which requires that the determinant of the \( 2 \times 2 \) matrix vanishes. The possible \( \omega^2 \) values are, therefore, solutions of
\[(\Omega^2 + 2\omega_0^2 + \omega^2)(\Omega^2 - \omega_0^2 + \omega^2) - 4\Omega^2\omega^2 = 0\]

or
\[
\left(\omega^2 + \frac{1}{2}\omega_0^2 - \Omega^2\right)^2 - \frac{1}{4}\omega_0^2(9\omega_0^2 - 8\Omega^2) = 0,
\]

which are two versions of the same second-degree polynomial in \(\omega^2\). This polynomial has two positive roots if (i) its value is positive for \(\omega^2 = 0\); (ii) its minimum is located at a positive \(\omega^2\) value; and (iii) the minimum is negative. Accordingly, we need

(i) \((\Omega^2 + 2\omega_0^2)(\Omega^2 - \omega_0^2) > 0\),

(ii) \(\Omega^2 - \frac{1}{2}\omega_0^2 > 0\),

(iii) \(\frac{1}{4}\omega_0^2(9\omega_0^2 - 8\Omega^2) > 0\).

It follows that the point mass stays near \(r = 0\) if

\[\omega_0^2 < \Omega^2 < \frac{9}{8}\omega_0^2.\]