Problem 1 (35=10+10+15 marks)
Isotropic harmonic oscillator: Point mass $m$ moves in the force field $F = -m\omega_0^2 r = -\nabla \frac{1}{2} m\omega_0^2 r^2$ with energy $E$ and angular momentum $l \neq 0$.

(a) By generalizing the familiar $x(t)$ of the one-dimensional harmonic oscillator, state $r(t)$ for initial position $r(t=0) = r_0$ and initial velocity $v(t=0) = v_0$. Express $E$ and $l$ in terms of $r_0$ and $v_0$.

(b) Show that the dyadic $D = vv + \omega_0^2 rr$ does not depend on time.

(c) Consider $(l \times r_0) \cdot r(t)$ and $(v_0 \times l) \cdot r(t)$ and use them to show that the plane orbit is an ellipse centered at $r = 0$. — Hint: $x(t), y(t)$ trace out a centered ellipse if $(x, y) A \begin{pmatrix} x \\ y \end{pmatrix} = 1$ with a positive, symmetric $2 \times 2$ matrix $A$.

Problem 2 (40=10+15+15 marks)
Point mass $m$ moves in the central-force field associated with the potential energy $V(r)$. The force is attractive, $V'(r) > 0$, and the motion is confined to the radial range $s_1 \leq r \leq s_2$. As usual the bounds are determined by the energy $E$ and the angular momentum $l$ of the orbit.

(a) Circular orbits ($s_1 = s_2$) have (i) smallest energy for given angular momentum and (ii) largest angular momentum for given energy. Explain why this is so.

(b) For the potential energy $V(r) = -\frac{A}{r(a+r)^2}$ with constants $A > 0$ and $a > 0$, one can have bound orbits for $E = 0$. For which values of $\kappa = |l|/m$ is this possible?

(c) Find the angular period of such an orbit with $E = 0$.

Problem 3 (25=15+10 marks)
A projectile with mass $m_1$ is scattered by a target of mass $m_2$, whereby a conservative line-of-sight force is acting. The target is at rest before the scattering. The scattering angle in the center-of-mass frame is denoted by $\theta$, that in the laboratory frame by $\Theta$.

(a) What is the range of possible $\Theta$ values when (i) $m_1 < m_2$, (ii) $m_1 = m_2$, (iii) $m_1 > m_2$?

(b) If the differential cross section in the center-of-mass frame is $\frac{d\sigma}{d\Omega} = f(\theta)$, what is the differential cross section observed in the laboratory frame when $m_1 = m_2$?