We have
\[ r(t) = r_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \]
as well as
\[ E = \frac{1}{2}mv_0^2 + \frac{1}{2}m\omega_0^2 r_0^2 \quad \text{and} \quad l = m r_0 \times v_0. \]

With \( \dot{v} = -\omega_0^2 r \) and \( \dot{r} = v \), the time derivative of \( D \) is
\[ \dot{D} = (\dot{v} v + v \dot{v}) + \omega_0^2 (\dot{r} r + r \dot{r}) = -\omega_0^2 (r v + v r) + \omega_0^2 (v r + r v) = 0. \]

Since \( (l \times r_0) \cdot r(t) = (l \times r_0) \cdot \frac{v_0}{\omega_0} \sin(\omega_0 t) = \frac{1}{m\omega_0} l^2 \sin(\omega_0 t) \)
and \( (v_0 \times l) \cdot r(t) = (v_0 \times l) \cdot r_0 \cos(\omega_0 t) = \frac{1}{m} l^2 \cos(\omega_0 t) \),
we have
\[ r(t) \cdot [a_1, a_2] \cdot r(t) = 1, \]
where
\[ a_1 = \frac{m\omega_0 l \times r_0}{l^2} \quad \text{and} \quad a_2 = \frac{mv_0 \times l}{l^2} \]
are non-parallel vectors in the plane of motion. If we choose that to be the \( xy \) plane, then \( (x y) A (x y) = 1 \) with a positive, symmetric \( 2 \times 2 \) matrix \( A \).

We have a circular orbit if \( E \) equals the minimum value of the effective potential energy \( V_{\text{eff}}(s) = V(s) + \frac{m\kappa^2}{2s^2} \), so this \( E \) is the smallest possible energy for the given \( \kappa \). In return, larger \( \kappa \) values are not possible for this \( E \), because then the minimum value of \( V_{\text{eff}}(s) \) would be larger, too.

Here, we have
\[ V_{\text{eff}}(s) = -\frac{A}{s(a+s)^2} + \frac{m\kappa^2}{2s^2} = \frac{1}{2s^2(a+s)^2} [m\kappa^2(a+s)^2 - 2As], \]
which is positive for sufficiently small distances \( s \) and for sufficiently large distances \( s \). Therefore, orbits with \( E = 0 \) are possible only if the minimum value of the factor \( \dots \) is negative. This is the case if
\[ \frac{m\kappa^2(s+a)}{A} = 2Aa - \frac{A^2}{m\kappa^2} < 0 \quad \text{or} \quad \kappa^2 < \frac{A}{2ma}. \]
(c) For

\[ V_{\text{eff}}(s) = -\frac{m\kappa^2}{2s^2(a+s)^2}(s_2-s)(s-s_1) \]

with

\[ s_1s_2 = a^2 \quad \text{and} \quad s_1 + s_2 = \frac{2A}{m\kappa^2} - 2a > 2a , \]

we get

\[ \Phi = 2 \int_{s_1}^{s_2} \frac{ds}{s} \kappa \sqrt{\frac{2}{m}s^2V_{\text{eff}}(s)} = 2 \int_{s_1}^{s_2} \frac{ds}{s} \frac{a+s}{\sqrt{(s_2-s)(s-s_1)}} \]

\[ = \frac{2\pi a}{\sqrt{s_1s_2}} + 2\pi = 4\pi \]

after using (5.3.21), (5.3.27), and (3.1.40).

3

(a) From Exercise 37, we know that

\[ \tan \Theta = \frac{\sin \theta}{m_1/m_2 + \cos \theta} . \]

(i) When \( m_1 < m_2 \), the denominator vanishes when \( \cos \theta = -m_1/m_2 > -1 \)

and is negative for larger \( \theta \) values. It follows that all values in the range

\[ 0 \leq \Theta \leq \pi \]

are possible.

(ii) When \( m_1 = m_2 \), we have \( \tan \Theta = \tan(\frac{1}{2}\theta) \) or \( \Theta = \frac{1}{2}\theta \), and all values in the range

\[ 0 \leq \Theta \leq \frac{1}{2}\pi \]

are possible.

(iii) When \( m_1 > m_2 \), the denominator is always positive, and \( \frac{d}{d\theta} \tan \Theta = 0 \)

when \( m_1 \cos \theta = -m_2 \), so that the largest possible value for \( \tan \Theta \)

is \( m_2/\sqrt{m_1^2 - m_2^2} \), and the possible \( \Theta \) values are from the range

\[ 0 \leq \Theta \leq \sin^{-1}(m_2/m_1) < \frac{1}{2}\pi . \]

(b) Corresponding cross sections are equal,

\[ \frac{d\sigma}{d\Omega} 2\pi d\theta \sin \theta = \left( \frac{d\sigma}{d\Omega} \right)_{\text{lab}} 2\pi d\Theta \sin \Theta ,\]

so that

\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{lab}} = f(\theta) \frac{\sin \theta}{\sin \Theta} \frac{d\theta}{d\Theta} \bigg|_{\theta = 2\Theta} = 4f(2\Theta) \cos \Theta . \]