According to Kepler’s Second Law, the time that the planet needs to cover the aphelion-side half of the ellipse is proportional to the area composed of the left half of the ellipse and the triangle. The time spent on the perihelion-side half of the ellipse is proportional to the area of the right half of the ellipse with the triangle removed.

The ellipse has area $\pi ab$, the triangle has area $\epsilon ab$. Therefore, it takes the fraction $\frac{1}{2} - \frac{\epsilon}{\pi}$ of the round-trip time to cover the perihelion-side half of the ellipse, and it takes the fraction $\frac{1}{2} + \frac{\epsilon}{\pi}$ to cover the aphelion-side half.

(a) When entering the ray is deflected by angle $\alpha - \beta$, and again by the same amount when exiting. Therefore, we have $\frac{1}{2}\theta = \alpha - \beta$ with $\sin \alpha = b/R = \sqrt{x}$ and $\sin \beta = \frac{3}{4}\sqrt{x}$, so that

$$y = \cos \left( \frac{1}{2} \theta \right) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \sqrt{1-x} \sqrt{1 - \frac{9x}{16} + \frac{3x}{4}}.$$  

We have $y \approx 1 - \frac{x}{32}$ for $0 \lesssim x \ll 1$ and $y = \frac{3}{4}$ for $x = 1$; in view of the $\sqrt{1-x}$ factor, the slope $\frac{dy}{dx}$ is infinite at $x = 1$. Here is the graph of $y(x)$:
With $b^2 = R^2 x$ and $\cos \theta = 2\cos \left(\frac{1}{2} \theta\right)^2 - 1 = 2y^2 - 1$ we have $db^2 = R^2 dx$ and $d \cos \theta = 4y dy$, so that

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \left| \frac{db^2}{d \cos \theta} \right| = \frac{R^2}{8y} \left| \frac{dx}{dy} \right| = -\frac{R^2}{8y} \left| \frac{dx}{dy} \right|,$$

where we recognize that $\frac{dx}{dy} < 0$. We express $x$ as a function of $y$,

$$x = 16 \frac{1 - y^2}{25 - 24y},$$

and differentiate to arrive at

$$\frac{d\sigma}{d\Omega} = \frac{4(4y - 3)(4 - 3y)}{y(25 - 24y)^2} R^2 \quad \text{with} \quad \frac{3}{4} \leq y = \cos \left(\frac{\theta}{2}\right) \leq 1.$$

With $d\Omega = d\phi \sin \theta \ d\theta = -d\phi \ d\cos \theta = -d\phi \ 4y \ dy$, we have

$$\sigma = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\phi \int_{3/4}^1 dy \ 4y \ \frac{d\sigma}{d\Omega}$$

$$= \int_0^{2\pi} d\phi \int_{3/4}^1 dy \ 4y \ R^2 \left( -\frac{dx}{dy} \right) = 2\pi R^2 \left( -x \right) \bigg|_{y=3/4} = \pi R^2.$$

As expected, the total cross section is the cross-sectional area of the water drop.

3

(a) We choose the coordinate system such that $r = 0$ is the position of the center-of-mass, so that $r_1 = -\frac{1}{2} a$ and $r_2 = \frac{1}{2} a$. According to Newton’s Shell Theorem, we then have the gravitational potential

$$-\frac{1}{2} GM \frac{1}{r + \frac{1}{2} a} - \frac{1}{2} GM \frac{1}{r - \frac{1}{2} a} = -G \int (dr') \frac{1}{2} M \delta(r' + \frac{1}{2} a) + \frac{1}{2} M \delta(r' - \frac{1}{2} a)$$

for points $r$ outside the two balls. It is as if we had two point masses $\frac{1}{2} M$ at $\pm \frac{1}{2} a$, with the as-if mass density

$$\rho(r') = \frac{1}{2} M \delta(r' + \frac{1}{2} a) + \frac{1}{2} M \delta(r' - \frac{1}{2} a).$$

The resulting quadrupole moment dyadic is

$$Q = \int (dr') \rho(r') \left( 3r' r' - r'^2 \mathbf{1} \right)$$

$$= 2 \times \frac{1}{2} M \left( 3 \left( \frac{1}{2} a \right) \left( \frac{1}{2} a \right) - \left( \frac{1}{2} a \right)^2 \mathbf{1} \right) = \frac{1}{4} M \left( 3 a a - a^2 \mathbf{1} \right).$$
At time $t = 0$, each ball is at distance $s(0) = \frac{1}{2}a$ from the center-of-mass that is half-way between the balls. At time $t = T$, the balls touch so that each ball is at distance $s(T) = R$ from the center-of-mass. Each ball is accelerated by the force $G(\frac{1}{2}M)^2/(2s)^2$ toward the center-of-mass, so that

$$\frac{1}{2}M\ddot{s} = -\frac{GM^2}{16s^2} \quad \text{or} \quad \ddot{s} = \frac{\partial}{\partial s} \frac{GM}{8s}.$$  

It follows that

$$s^2 - \frac{GM}{4s} = -\frac{GM}{2a} = \text{constant},$$

with the value of this constant determined by $s(0) = \frac{1}{2}a$ and $\dot{s}(0) = 0$. Since $\dot{s}(t) < 0$ for $t > 0$, we have

$$\dot{s} = \frac{ds}{dt} = -\sqrt{\frac{GM}{2a}} \sqrt{\frac{a/2 - s}{s}}$$

and

$$T = \int_0^T dt = \int \frac{2a}{GM} \int_R^{a/2} ds \sqrt{\frac{s}{a/2 - s}} = \sqrt{\frac{a^3/2}{GM}} \int_{2R/a}^1 dx \sqrt{\frac{x}{1-x}}.$$  

For $a \gg R$, the $x$ integral gives $\frac{1}{2}\pi$, and $T \simeq \pi \sqrt{\frac{(a/2)^3}{GM}}$ follows.
Along the path specified by \( y(x) \), it takes time
\[
T[y] = \int_0^a dx \sqrt{\frac{1 + y'(x)^2}{2gy(x)}
\]
to cover the path-length
\[
S[y] = \int_0^a dx \sqrt{1 + y'(x)^2},
\]
and the average speed is \( S[y]/T[y] \).

(a) For the straight-line path, we have \( y(x) = bx/a \), which gives
\[
S = \sqrt{a^2 + b^2}
\]
and \( T = \sqrt{a^2 + b^2} \sqrt{2/(gb)} \); the average speed is
\[
\sqrt{\frac{1}{2} gb} = \sqrt{gR \sin \frac{\phi_0}{2}}.
\]
For the brachistochrone, we have \((dx)^2 + (dy)^2 = 2Ry(d\phi)^2 = (2R \sin \frac{\phi}{2})^2 (d\phi)^2\), so that
\[
S = \int_0^{\phi_0} d\phi 2R \sin \frac{\phi}{2} = 4R \left( 1 - \cos \frac{\phi_0}{2} \right) = 8R \left( \sin \frac{\phi_0}{4} \right)^2
\]
is the path-length and
\[
T = \int_0^{\phi_0} d\phi \sqrt{\frac{2Ry}{2gy}} = \sqrt{\frac{R}{g} \phi_0}
\]
is the travel time; the average speed is \( \sqrt[4]{gR} \frac{8}{\phi_0} \left( \sin \frac{\phi_0}{4} \right)^2 \).
The ratio of the two average speeds is
\[
\frac{\text{brachistochrone}}{\text{straight line}} = \frac{\frac{8}{\phi_0} \left( \sin \frac{\phi_0}{4} \right)^2}{2 \sin \frac{\phi_0}{4} \cos \frac{\phi_0}{4}} = \frac{\tan(\phi_0/4)}{\phi_0/4} > 1,
\]
since \( 0 < \frac{1}{4} \phi_0 < \frac{1}{2} \pi \).

(b) As observed in (a), the average speed along a straight-line path with height difference \( B \) is \( \sqrt{\frac{1}{2} gB} \) if the speed is zero at the upper end, and it will be larger than that if the speed at the upper end is nonzero. We can choose a path that goes on a straight line from \((0,0)\) to an intermediate point \((a',B)\) with \( B > b \) and then on another straight line to \((a,b)\), and so get an average speed of \( \sqrt{\frac{1}{2} gB} \) or more. Since \( B \) can be as large as we like, the average speed can exceed any bound. Conclusion: There is no path for which the average speed is largest.