According to Section 4.5 of the lecture notes, we have

\[
\frac{d}{dt} R = \frac{1}{M} P_{\text{tot}},
\]

\[
\frac{d}{dt} P_{\text{tot}} = \sum_{j=1}^{J} F_j^{(\text{ext})} = M g,
\]

\[
\frac{d}{dt} E_{\text{tot}} = \sum_{j=1}^{J} v_j \cdot F_j^{(\text{ext})} = P_{\text{tot}} \cdot g,
\]

\[
\frac{d}{dt} L_{\text{tot}} = \sum_{j=1}^{J} r_j \times F_j^{(\text{ext})} = M R \times g.
\]

These are solved by

\[
R(t) = R_0 + \frac{P_0}{M} (t - t_0) + \frac{1}{2} g(t - t_0)^2,
\]

\[
P_{\text{tot}}(t) = P_0 + M g(t - t_0),
\]

\[
E_{\text{tot}}(t) = E_0 + P_0 \cdot g(t - t_0) + \frac{1}{2} M g^2(t - t_0)^2,
\]

\[
L_{\text{tot}}(t) = L_0 + M R_0 \times g(t - t_0) + \frac{1}{2} P_0 \times g(t - t_0)^2.
\]

Scattering occurs only when \( b < R \). For \( b < R \), then, we denote the angle of incidence by \( \alpha \). It is related to the impact parameter by \( b = R \sin \alpha \) and to the scattering angle by \( \theta = \pi - 2\alpha \). Therefore,

\[
\cos \theta = -\cos(2\alpha) = 2(\sin \alpha)^2 - 1 = 2(b/R)^2 - 1,
\]

which implies

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} \frac{db^2}{d\cos \theta} = \frac{1}{4} R^2
\]

for the differential cross section and \( \sigma = \pi R^2 \) for the total cross section.
According to Section 7.3 of the lecture notes, we have

\[ x = R(\phi - \sin \phi), \]
\[ y = R(1 - \cos \phi) \]

for the brachistochrone, here with \( 0 \leq \phi \leq 2\pi \) and \( R = a/(2\pi) \). With

\[ ds = \sqrt{(dx)^2 + (dy)^2} = Rd\phi \sqrt{2 - 2\cos \phi} = 2Rd\phi \sin \frac{\phi}{2} \]

and

\[ v = \sqrt{2gy} = 2\sqrt{gR} \sin \frac{\phi}{2} \]

we get

\[ T = \int \frac{ds}{v} = \int_0^{2\pi} d\phi \frac{R}{\sqrt{gR}} = 2\pi \sqrt{R/g} \]

for the duration and

\[ S = \int ds = 2R \int_0^{2\pi} d\phi \sin \frac{\phi}{2} = 8R \]

for the distance covered. Their ratio is the average speed,

\[ \text{average speed} = \frac{S}{T} = \frac{4}{\pi} \sqrt{gR} = \frac{4}{\pi} \sqrt{\frac{ga}{2\pi}}. \]

(a) We have the Lagrange function

\[ L(t, x, \dot{x}) = \frac{m}{2} \dot{x}^2 - \frac{k}{2} \left( \sqrt{x^2 + a^2} - a \right)^2; \]

it has no parametric \( t \) dependence. The energy

\[ E = \frac{m}{2} \dot{x}^2 + \frac{k}{2} \left( \sqrt{x^2 + a^2} - a \right)^2 \]

is conserved. The equation of motion is

\[ \frac{d}{dt} m\dot{x} = -k \left( \sqrt{x^2 + a^2} - a \right) \frac{x}{\sqrt{x^2 + a^2}} = -kx + \frac{kax}{\sqrt{x^2 + a^2}}. \]
Since $\dot{x} = a\dot{\vartheta} \cosh \vartheta$ and $\sqrt{x^2 + a^2} = a \cosh \vartheta$, we have the Lagrange function

$$L(t, \vartheta, \dot{\vartheta}) = \frac{ma^2}{2} (\dot{\vartheta} \cosh \vartheta)^2 - \frac{k a^2}{2} (\cosh \vartheta - 1)^2.$$ 

This gives the equation of motion

$$m \frac{d}{dt} [\dot{\vartheta} (\cosh \vartheta)^2] = m \dot{\vartheta}^2 \cosh \vartheta \sinh \vartheta - k (\cosh \vartheta - 1) \sinh \vartheta$$

or

$$\ddot{\vartheta} (\cosh \vartheta)^2 + \dot{\vartheta}^2 \cosh \vartheta \sinh \vartheta = -\frac{k}{m} (\cosh \vartheta - 1) \sinh \vartheta.$$

For $|x| \ll a$ we have $\sqrt{x^2 + a^2} - a = \frac{x^2}{2a} + \cdots$ and get the approximate Lagrange function

$$L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - \frac{k x^4}{8a^2}$$

and the equation of motion

$$m \ddot{x} = -\frac{k x^3}{2a^2}.$$ 

Since $x = a \vartheta$ here, we also have $L(\vartheta, \dot{\vartheta}) = \frac{ma^2}{2} \dot{\vartheta}^2 - \frac{k a^2 \dot{\vartheta}^4}{8}$ and $\ddot{\vartheta} = -\frac{k}{2m} \vartheta^3$. 