1. Rayleigh scattering differs from Thomson scattering only by the frequency dependent factor, so

\[
\frac{d\sigma}{d\Omega}_{\text{Ray}} \sim \frac{d\sigma}{d\Omega}_{\text{Thom}} \frac{1}{\omega^2 - \omega^2_{1,2} - i\gamma \omega^2}
\]

\[
= \frac{\nu^2}{2} \left( 1 + (\cos \theta)^2 \right) \frac{\omega^4}{(\omega^2 - \omega^2_{1,2})^2 + (\gamma \omega)^2}.
\]

2. (a) The angle of incidence \(\theta\) must be less than the critical angle for total internal reflection,

\[\sin \theta < \frac{1}{n} = \sin \theta_c.\]

The angle \(\theta_c\) between \(\vec{V}\) of the electron and the normal surface vector \(\vec{e}_z\) is denoted by \(\vec{\theta}\), so that Cherenkov radiation can get across the surface if

\[\vec{\theta} = \Theta + \alpha < \Theta + \alpha_0\]

where \(\cos \Theta = \frac{c}{nu} \) (Cherenkov) and \(\sin \alpha_0 = \frac{1}{n} \) (Shell).
(b) We would need $\theta + \phi_0 > \frac{\pi}{2}$, but in fact we have

$$\cos(\theta + \phi_0) = \cos \theta \cos \phi_0 - \sin \theta \sin \phi_0$$

$$= \frac{c}{\mu_0} \left( 1 - \frac{1}{n^2} \right) - \frac{1}{n} \sqrt{1 - \left( \frac{c}{n} \right)^2}$$

$$= \frac{1}{n^2} \left( (nc)^2 - c^2 \right) - \sqrt{(nv)^2 - c^2} > 0$$

which tells us that $\theta + \phi_0 < \frac{\pi}{2}$. Answer: No.

[3] (a) The retarded time (= emission time corrected for internal retardation) for the $l$th ring antenna ($l = 0, \pm 1, \pm 2, \ldots, \pm N$) is

$$t_r = t - \frac{r}{c} + \frac{1}{c} \mathbf{n} \cdot \mathbf{r'} - \frac{1}{c} \mathbf{n} \cdot \mathbf{D} \mathbf{E}_z$$

central antenna with $l=0$
displacement of the $l$th antenna

$$= \left( t - \frac{r}{c} - \frac{l}{c} \cos \theta \right) + \frac{1}{c} \mathbf{n} \cdot \mathbf{r'}$$

Effectively we have the replacement

$$t_e = t - \frac{r}{c} \rightarrow t_e - \frac{l}{c} \cos \theta.$$

The integrated time derivative of the current lies a time dependence given by

$$\cos \left( \omega t_0 + \phi \right)$$
Question... 3/5

Write answers on this side of the paper only.

For the central antenna \((l=0)\) whereby \(x\) is some irrelevant phase that will not survive the averaging over one period. Accordingly,

\[
\left( \frac{dP}{d\theta} \right)_N = \left( \frac{dP}{d\theta} \right)_1 \left( \sum_{e=-M}^{M} a_0/(\omega t_e + \alpha - \frac{1}{c} l \omega D \cos \theta) \right)^2
\]

For all \(N\) central only

Where the sum is a geometric summation,

\[
\sum_{e=-M}^{M} a_0/(\omega t_e + \alpha - \frac{1}{c} l \omega D \cos \theta) = \Re \left[ e^{i(\omega t_0 + \alpha)} \sum_{e=-M}^{M} \frac{-i l \omega D}{c} \cos \theta \right]
\]

\[
= \cos (\omega t_0 + \alpha) \frac{\sin(2\pi t_1 \omega D / 2c \cos \theta)}{\sin(\omega D / 2c \cos \theta)}
\]

With \(N = 2M + 1\) and \(\omega D / 2c = \pi \frac{P}{x}\), then,

\[
\left( \frac{dP}{d\theta} \right)_N = \left( \frac{dP}{d\theta} \right)_1 \left( \frac{\sin(\pi \frac{N \omega D}{x} \cos \theta)}{\sin(\pi \frac{D}{x} \cos \theta)} \right)^2
\]

\[
= \frac{\pi}{2} \frac{l_0^2}{c} \frac{a_0}{(\frac{l_0}{c})^2} \sum_{e=0}^{M} [a_0 (\frac{\omega D}{c} \sin \theta)]^2 \left( \frac{\sin(\pi \frac{N \omega D}{x} \cos \theta)}{\sin(\pi \frac{D}{x} \cos \theta)} \right)^2
\]

One central antenna only

Interference of all \(N\) antennas
(b) The interference enhances the radiation emitted into the $xy$ plane where $\cos \theta = 0$, and the interference term equals $N^2$. There is destructive interference for those directions with 

$$N \frac{\lambda}{D} \sin \theta = \pm 1, \pm 2, \ldots$$

We get additional suppression of radiation in the $z$ direction (on top of the vanishing of $J_1(\omega \cdot \mathbf{D})$ for $\sin \theta = 0$) when $N \frac{\lambda}{D}$ is integer.

4) In the relativistic energy-loss formula

$$\frac{dE}{dt} = \frac{2e^2}{3c^3} \left[1 - \left(\frac{v(t)}{c}\right)^2\right]^{-3} \left[\left(\frac{dv}{dt}\right)^2 - \left(\frac{v^2}{c^2} + \frac{dv}{dt}\right)^2\right]$$

we insert

$$\frac{d}{dt} \frac{\mathbf{v}}{c} = -\frac{\mathbf{v}}{T}, \quad \mathbf{v}(t) = \left(1 - \frac{t}{T}\right) \mathbf{v}_0,$$

valid while the stopping process lasts. This gives the total radiated energy

$$E_{\text{rad}} = \frac{2e^2}{3c^3} \int_0^T dt \left[1 - \left(\frac{v_0}{c}\right)^2 \left(1 - \frac{t}{T}\right)^2\right]^{-3} \left(\frac{v_0}{c}\right)^2 \left(\frac{v_0}{c}\right)^2 \left(\frac{v_0}{c}\right)^2$$

or, after the substitution

$$\frac{v_0}{c} \left(1 - \frac{t}{T}\right) = \tanh \delta,$$
\[ E\theta = \frac{2}{3} \frac{e^2}{c^2 T} \left( \frac{v_0}{c} \right)^2 \int_0^\infty \sin^3 (\cos \theta) \, d\theta \]

where \( \theta \) is the rapidity associated with \( v_0 \), \( v_0 = c \tanh \theta \).

Using \((\cos \theta)^3 = \frac{1}{2} + \frac{1}{2} \cosh (2\theta)\) twice, we get

\[ E\theta = \frac{2}{3} \frac{e^2}{c^2 T} \left( \frac{v_0}{c} \right)^2 \left[ \frac{3}{8} \theta + \frac{3}{4} \sinh (2\theta) + \frac{1}{5} \cosh (4\theta) \right]^{\frac{3}{2}} \]

\[ = \frac{2}{3} \frac{e^2}{c^2 T} \left( \frac{v_0}{c} \right)^2 \left[ \frac{3}{8} \theta + \frac{3}{8} \sinh (2\theta) \cosh (2\theta) + \frac{1}{4} \cosh (4\theta) \right] \]

with \( \beta_0 = \frac{v_0}{c} \), \( \gamma_0 = \sqrt{1 - \beta_0^2} = \cosh \theta \).

(b) When \( \beta_0 \ll 1, \gamma_0 \gg 1 \), the last term in the parentheses dominates the others, and we have

\[ E\theta \approx \frac{2}{3} \frac{e^2}{c^2 T} \frac{1}{4} \gamma_0^4 = \frac{1}{6} \frac{e^2}{c^2 T} \gamma_0^4 \]

This can also be derived by realizing that \((\cosh \theta)^4 \approx \left( \frac{1}{2} e^{\gamma_0^2} \right)^4 = \frac{1}{4 \gamma_0^2} \left( \frac{1}{2} e^{\gamma_0^2} \right)^4\) in the integral at the top of the page, under these circumstances.