The Problem (100 marks)

Consider this ring antenna: a thin wire in the shape of a circle with radius $a$ in the $x, y$ plane, centered at $\vec{r} = 0$, that carries a periodic current $I \cos(\omega t)$. The electric current density is

$$\vec{j}(\vec{r}, t) = I \cos(\omega t) \, \delta(r - a) \, \delta(z) \, \vec{e}_z \times \frac{\vec{r}}{r},$$

and we are interested in $\frac{dP}{d\Omega}$, the angular distribution of the radiated power, averaged over one period.

(a) Explain why $\omega \ll c$ means a “small” antenna? What is $\frac{dP}{d\Omega}$ in the small-antenna limit? What is the corresponding total power $P = \int d\Omega \frac{dP}{d\Omega}$?

(b) Find $\frac{dP}{d\Omega}$ without any assumption about the value of $\omega/c$.

(c) Sketch the pattern of the angular distribution of the radiation emitted by the ring antenna for $\omega \ll c$ and $\omega \gg c$ and describe how the patterns differ. The plot below of the Bessel functions $J_0(x)$ and $J_1(x)$ could be helpful.