1. The Euler-Lagrange equation

\[ 2 \left( \frac{d}{dx} \right)^2 f(x) = \frac{15}{2} \frac{f'(x)^2}{\sqrt{x}} \]

is solved by \( f(x) = x^{-3/4} \) which obeys the constraints \( f(1) = 1, f(\infty) = 0 \). The integral has the value

\[ \int_1^\infty \left( \frac{9}{4} + \frac{5}{2} \right) x^{-5} = \frac{19}{16} \]

2. In view of

\[
\begin{align*}
B^2 : \ 3 & \mapsto -3 \\
B^3 : \ 3 & \mapsto -i3 \\
B^4 : \ 3 & \mapsto 3; \quad B^4 = E = B B^3 = B^2 B^2 = B^2 B \\
A B : \ 3 & \mapsto -i3^* \\
A B^2 : \ 3 & \mapsto -3^* \\
A B^3 : \ 3 & \mapsto i3^* \\
A B^4 : \ 3 & \mapsto 3^* \\
B A & = A B^3 \\
B^2 A & = B A B^3 = A B^2 \\
A B A & = A^2 B^3 = B^3 \\
B A B & = A B^4 = A \\
B A B^2 & = A B \\
B^3 A & = B A B^2 = A B
\end{align*}
\]

the completed table is...
(b) Subgroups with two elements consist of

E and either A, or \( B^2 \), or \( AB \), or \( AB^2 \),

or \( AB^3 \).

Subgroups with four elements are

\( \{ E, B, B^2, B^3 \} \), the cyclic group,

and \( \{ E, A, B^2, AB \} \), the vierergruppe,

and \( \{ E, B^2, AB, AB^3 \} \), the vierergruppe again.

(c) All are abelian.
The Laplace transform $F(s) = \int_0^\infty dt e^{-st} f(t)$ obeys

$$F(s) + \frac{d}{ds}(sF(s) - f(0)) = 2F(s)^2$$

or

$$s \frac{d}{ds} F(s) + 2F(s) = 2F(s)^2$$

and

$$\int_0^\infty ds F(s) = \pi.$$ 

The differential equation has the general solution

$$F(s) = \frac{a}{a+s^2}$$

and we need $a > 0$ for the integral to converge. Since

$$\int_0^\infty ds \frac{a}{a+s^2} = \frac{\pi}{2} \sqrt{a}$$

we find $a = 4$, so that

$$F(s) = 2 \frac{2}{s^2+2^2}$$

and $f(t) = 2 \sin(2t)$. 
(a) 
\[ f(z)^2 + 1 = (\sin \theta \cos \phi)^2 + (\cos \theta)^2 - (\cos \theta \sin \phi)^2 + (\sin \theta \sin \phi)^2 + 2i \sin \theta \cos \theta \cos \phi \sin \phi \]
\[ = (\cos \theta \cos \phi + i \sin \theta \sin \phi)^2 = z^2. \]

(b) In the limit \( z \to 1 \) and \( 0 < 1 \), we have \( \theta \to 0 \), \( \cos \phi \to \infty \), \( \sin \phi \to \pm \sqrt{1 - \infty^2} \), so that
\[ f(z_0 + i\varepsilon) \to i \sqrt{1 - x_0^2}, \]
\[ f(z_0 - i\varepsilon) \to -i \sqrt{1 - x_0^2}, \]
and
\[ f(z_0 + i\varepsilon) - f(z_0 - i\varepsilon) \to 2i \sqrt{1 - x_0^2}. \]

(c) Here \( dz = d\phi \) \( (-\cos \theta \sin \phi + i \sin \theta \cos \phi) \)
\[ = i d\phi \quad f(z) \]
so that
\[ dz \cdot f(z) = i d\phi \cdot f(z)^2 \]
\[ = i d\phi \left[ -\frac{1}{2} + \frac{1}{2} \cosh (2\theta) \cos (2\phi) + \frac{i}{2} \sinh (2\theta) \sin (2\phi) \right] \]
and
\[ \oint_C dz \cdot f(z) = -i \pi \]
follows.
(d) We have

\[ f(z) = 3 \left(1 - \frac{1}{z^2}\right)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} z^2 \left(-\frac{1}{z^2}\right)^n \]

\[ = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n z^{1-2n} \]

\[ = 3 - \frac{1}{2z} + \frac{1}{8z^3} + \ldots \]

so that the residue is \(-\frac{1}{2}\) and the integral \(\text{Res}(f, z)\) is \((2\pi i)(-\frac{1}{2}) = -\pi i\), indeed.