Since we want to evaluate the smallest value (or extremum value) that is possible for the integral, we let

\[
\left( \frac{d}{dx} y(x) \right)^2 + [y(x)]^3 = F, \tag{4}
\]

which can also be expressed as

\[
y'^2 + y^3 = F, \tag{5}
\]

and hence applying the identity,

\[
F - y' \frac{\partial F}{\partial y'} = \text{constant}. \tag{6}
\]

Substituting (5) into (6):

\[
y'^2 + y^3 - y' (2y') = \text{constant}. \tag{7}
\]

However, as \( x \to \infty \), \( y' \) and \( y \) are 0, therefore the constant is 0 too.

Simplifying,

\[
y'^2 = y^3. \tag{8}
\]

Therefore, we get a differential equation in which \( y' \) must be negative because \( y \) decreases as \( x \) increases:

\[
\frac{dy}{dx} = -y^3. \tag{9}
\]

Bringing the \( y \) term over,

\[
\int -y^{-\frac{3}{2}} dy = \int dx. \tag{10}
\]

Solving (10) gives,

\[
2y^{-\frac{1}{2}} = x + c, \tag{11}
\]

where \( c \) is a constant to be determined by boundary conditions.

When \( x=0, y=1 \), substituting this into (11):

\[
c = 2.
\]

Therefore, squaring (11) and using the fact that \( c = 2 \), we get:
\[ y = \frac{4}{(x + 2)^2} \]  

(12)

Substituting (12) into the integral below,

\[ \int_0^\infty dx \left( \left[ \frac{d}{dx} y(x) \right]^2 + [y(x)]^3 \right), \]

we get

\[ \int_0^\infty dx \left( \left[ -\frac{8}{(x + 2)^3} \right]^2 + \left[ \frac{4}{(x + 2)^2} \right]^3 \right), \]

(13)

which can be evaluated as

\[ \int_0^\infty dx \left( \frac{64}{(x + 2)^6} + \frac{64}{(x + 2)^6} \right). \]

(14)

Evaluating (14), it becomes

\[ -\frac{128}{5(x + 2)^5} \bigg|_0^\infty = \frac{4}{5}. \]

(15)

Hence the smallest value for the integral is \(\frac{4}{5}\).

In Cartesian coordinates, \( L(x, y, z, \dot{z}) = \frac{1}{2} m(x^2 + y^2 + z^2) - mgz \).

Using the given parameterization,

\[ z = \sqrt{x^2 + y^2 + \dot{a}^2} \]

\[ = \sqrt{\dot{a}^2 \sinh^2 \xi \cos^2 \varphi + \dot{a}^2 \sinh^2 \xi \sin^2 \varphi + \dot{a}^2} \]

\[ = \dot{a} \sqrt{\sinh^2 \xi + 1} \]

\[ = \dot{a} \cosh \xi. \]

Then,

\[ \dot{x} = \dot{a} \cosh \xi \cos \varphi \xi - \dot{a} \sinh \xi \sin \varphi \xi, \]

(16)

\[ \dot{y} = \dot{a} \cosh \xi \sin \varphi \xi + \dot{a} \sinh \xi \cos \varphi \xi, \]

(17)

\[ \dot{z} = \dot{a} \sinh \xi \xi, \]

(18)
\[ x^2 + y^2 + z^2 = a^2 \cosh^2 \zeta \cos^2 \phi \quad \zeta^2 + a^2 \sinh^2 \zeta \sin^2 \phi \quad \dot{\phi}^2 \]
\[ + a^2 \cosh^2 \zeta \sin^2 \phi \quad \dot{\zeta}^2 + a^2 \sinh^2 \zeta \cos^2 \phi \quad \dot{\phi}^2 \]
\[ + a^2 \sinh^2 \zeta \dot{\zeta}^2 \]
\[ = a^2 (\cosh^2 \zeta + \sinh^2 \zeta) \dot{\zeta}^2 + a^2 \sinh^2 \zeta \dot{\phi}^2 . \]

Note that the terms containing \((\zeta \dot{\phi})\) in (19) cancel out.

So in the new coordinates,

\[ L(\zeta, \dot{\zeta}, \dot{\phi}) = \frac{1}{2} m a^2 [(\cosh^2 \zeta + \sinh^2 \zeta) \dot{\zeta}^2 + \sinh^2 \zeta \dot{\phi}^2] - mga \cosh \zeta. \]

To find Hamilton function, we need \(p_\zeta\) and \(p_\phi\),

\[ p_\zeta = \frac{\partial L}{\partial \dot{\zeta}} = m a^2 (\cosh^2 \zeta + \sinh^2 \zeta) \dot{\zeta}, \]

\[ \dot{\zeta} = \frac{p_\zeta}{ma^2 (\cosh^2 \zeta + \sinh^2 \zeta)}, \]

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \sinh^2 \zeta \dot{\phi}, \]

\[ \dot{\phi} = \frac{p_\phi}{ma^2 \sinh^2 \zeta}. \]

Substituting \(\dot{\zeta}\) and \(\dot{\phi}\) into

\[ H = p_\zeta \dot{\zeta} + p_\phi \dot{\phi} - L, \]

Finally,

\[ H(\zeta, p_\zeta, p_\phi) = \frac{p_\zeta^2}{2ma^2 (\cosh^2 \zeta + \sinh^2 \zeta)} + \frac{p_\phi^2}{2ma^2 \sinh^2 \zeta} + mga \cosh \zeta. \]

Equations of motion:

\[ \frac{dp_\phi}{dt} = -\frac{\partial H}{\partial \phi} = 0, \]
\[
\frac{dp_\zeta}{dt} = \frac{\partial H}{\partial \zeta} = \frac{4 \sinh \zeta \cosh \zeta}{2m\alpha^2 (\cosh^2 \zeta + \sinh^2 \zeta)} + \frac{p_\zeta^2 (2 \text{cosech}^2 \zeta \coth \zeta)}{2m^2} - mga \sinh \zeta
\]
\[
= \frac{p_\zeta^2}{m\alpha^2 (\cosh^2 \zeta + \sinh^2 \zeta)} + \frac{p_\psi^2 \text{cosech}^2 \zeta \coth \zeta}{m^2} - mga \sinh \zeta
\]
(28)

\[
\frac{d \phi}{dt} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\alpha^2 \sinh^2 \zeta}
\]
(29)

\[
\frac{d \zeta}{dt} = \frac{\partial H}{\partial p_\zeta} = \frac{p_\zeta}{ma^2 (\cosh^2 \zeta + \sinh^2 \zeta)}
\]
(30)

3. We know that \( \frac{d}{dt} \rho = 0 \), so by the general form of Hamilton’s equation of motion,
\[
\frac{d}{dt} \rho = \frac{d}{dt} \rho(x(t), p(t), t) = \frac{\partial \rho}{\partial x} \frac{dx(t)}{dt} + \frac{\partial \rho}{\partial p} \frac{dp(t)}{dt} + \frac{\partial \rho}{\partial t} = \{\rho, H\} + \frac{\partial \rho}{\partial t} = 0.
\]
(31)

Hence, by Liouville’s equation,
\[
\frac{\partial \rho}{\partial t} = -\{\rho, H\} = \frac{\partial \rho}{\partial p} \frac{\partial H}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial H}{\partial p}.
\]
(32)

Taking time derivative of the function in question,
\[
\frac{d}{dt} \left( \int dx dp \rho(x, p, t) \right) = \int dx dp \frac{d}{dt} \rho(x, p, t)
\]
(33)

Substituting equation (32) into equation (33),
\[
\frac{d}{dt} \left( \int dx dp \rho(x, p, t) \right) = \int dx dp \left( \frac{\partial \rho}{\partial p} \frac{\partial H}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial H}{\partial p} \right)
\]
\[
= \int dx dp \frac{\partial \rho}{\partial p} \frac{\partial H}{\partial x} - \int dx dp \frac{\partial \rho}{\partial x} \frac{\partial H}{\partial p}
\]
\[
= \int dx \left( [\rho \frac{\partial H}{\partial x}]_{p=\infty}^p - \int dp \rho \frac{\partial^2 H}{\partial p \partial x} \right) - \int dp \left( [\rho \frac{\partial H}{\partial p}]_{x=\infty}^x \int dx \rho \frac{\partial^2 H}{\partial x \partial p} \right)
\]

\[
= - \int dx dp \rho \frac{\partial^2 H}{\partial p \partial x} + \int dp dx \rho \frac{\partial^2 H}{\partial x \partial p}
\]

\[
= \int dx dp \rho \left(-\frac{\partial^2 H}{\partial p \partial x} + \frac{\partial^2 H}{\partial x \partial p} \right) = 0.
\]

Hence, it can be seen that \( \int dx \ dp \ \rho(x, p, t) \) is independent of time, \( t \), and that integration covers the whole phase space.

For iso-\( z \) lines, we have the following differential equations,

\[
\frac{dx}{x + z} = \frac{dy}{2}
\]  

(34)

Next, we evaluate in terms of \( x, y, z \).

\[
2 \ln|x + z| - y = \text{constant}.
\]  

(35)

The fact that we can integrate while treating \( z \) as a constant is because we are using the iso-\( z \) lines. Therefore simplifying (35):

\[
(x + z)e^{-\frac{1}{2}y} = \text{constant}.
\]  

(36)

We now let \( z(x, y) = u((x + z)e^{-\frac{1}{2}y}) \), where \( u \) is an arbitrary function.

Using the fact that \( z(x, 0) = x \), we get,

\[
z(x, 0) = x = u(2x).
\]  

(37)

Upon letting \( 2x = a \), (37) will now become

\[
\frac{1}{2} a = u(a).
\]  

(38)

Now by letting \( a = (x + z)e^{-\frac{1}{2}y} \), we can get the solution for \( z(x, y) \),

\[
u \left((x + z)e^{-\frac{1}{2}y}\right) = \frac{1}{2} (x + z)e^{-\frac{1}{2}y} = z(x, y),
\]  

(39)

\[
z = \frac{x}{2e^{2y} - 1}.
\]  

(40)
Now, to verify whether this solution is correct or not, we substitute (40) into

\[
\left[(x + z) \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial y}\right] z = 0 .
\]

First, we find

\[
\frac{\partial z}{\partial x} = \frac{1}{\frac{1}{2e^{2y}} - 1} = \frac{z}{x},
\]

\[
\frac{\partial z}{\partial y} = -\frac{xe^{\frac{1}{2y}}}{(2e^{\frac{1}{2y}} - 1)^2} = -\frac{z^2e^{\frac{1}{2y}}}{x}.
\]

We now try to evaluate the 2 terms in the qIPDE replacing \(x\) using (40),

\[
(x + z) \frac{\partial z}{\partial x} = \left(z + \frac{z^2}{x}\right) = \left(z + \frac{z}{2e^{\frac{1}{2y}} - 1}\right),
\]

\[
2 \frac{\partial z}{\partial y} = -\frac{2z^2e^{\frac{1}{2y}}}{x} = -\frac{2ze^{\frac{1}{2y}}}{2e^{\frac{1}{2y}} - 1}.
\]

Now, by summing (43) and (44), we should get 0, if not \(z\) will be incorrect,

\[
(43 + 44):
\]

\[
z + \frac{z}{2e^{\frac{1}{2y}} - 1} - \frac{2ze^{\frac{1}{2y}}}{2e^{\frac{1}{2y}} - 1} = \frac{2ze^{\frac{1}{2y}}}{2e^{\frac{1}{2y}} - 1} - z + z - 2ze^{\frac{1}{2y}} = 0 .
\]

Hence, we verified that

\[
z = \frac{x}{2e^{\frac{1}{2y}} - 1}.
\]