Question \textbf{Exam/1}

Write answers on this side of the paper only.

\textbf{(a)} (i) Generally, \( \frac{\partial}{\partial t} \rho + \{ \rho, H \} = 0; \)
here: \( \left( \frac{\partial}{\partial t} + \left( \frac{\rho}{m} - \gamma x \right) \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial \rho} \right) \rho = 0. \)

(ii) Equations for the characteristic curves are
\[ dx = \left( \frac{\rho}{m} - \gamma x \right) \, dt, \]
\[ dp = \gamma \rho \, dt. \]
Solved by
\[ p(t) = p_0 e^{\gamma t}, \]
\[ x(t) = x_0 e^{-\gamma t} + \frac{p_0}{2ym} (e^{\gamma t} - e^{-\gamma t}), \]
so that the characteristic curves are
\[ p_0 = p e^{-\gamma t} = \text{const}, \]
\[ x_0 = x e^{\gamma t} - \frac{p}{2ym} (e^{\gamma t} - e^{-\gamma t}) = \text{const}. \]
This then gives
\[ \rho(t, x, p) = p_0 \left( x e^{\gamma t} - \frac{p}{2ym} (e^{\gamma t} - e^{-\gamma t}), p e^{-\gamma t} \right). \]

(iii) \( \int dx dp \, \rho(t, x, p) = \int dx' dp' \, \rho_0 (x', p') \)
after substitution
\[ x' = x e^{\gamma t} - \frac{p}{2ym} (e^{\gamma t} - e^{-\gamma t}), \]
\[ p' = p e^{\gamma t}, \]
for which \( dx' dp' = dx dp. \)
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\( \int_0^1 \left( \frac{dy}{dx} x^2 y' \right) = 0 \) or

\[
\int_{-1}^1 \frac{d}{dx} \frac{dy}{dx} \frac{d}{dx} (xy') = 0
\]

Thus we want

\[
\int_0^1 dx \frac{dy}{dx} \frac{d}{dx} (xy') = 0
\]

with the constraint

\[
\int_0^1 dx x \frac{dy}{dx} = 0
\]

so that

\[
\frac{d}{dx} (xy') = 2x
\]

with Lagrange multiplier \( \lambda \). This implies first

\[ y' = \frac{1}{2} \lambda x \]

then

\[ y(x) = \frac{\lambda}{4} (x^2 - 1) \]

where \( y(1) = 0 \) and \( y'(0) = 0 \) are taken into account. The constraint

\[ 1 = \int_0^1 dx x y(x) = \frac{\lambda}{4} (\frac{1}{2} - \frac{1}{2}) = -\frac{\lambda}{16} \]

gives \( \lambda = -16 \) and \( y(x) = 4 - 4x^2 \),

so that the looked-for minimal value is

\[
\int_0^1 dx x (-8x)^2 = 16.
\]
Question

Write answers on this side of the paper only.

12 (a) Neutral element: \( e = (0, 0, 1) \).

Inverse element: \( \tilde{g} = (-a, -b, u^* e^{i\alpha}) \).

Composition is associative:

\[
\tilde{g}_1 \tilde{g}_2 \tilde{g}_3 = (a_1 + a_2 + a_3, b_1 + b_2 + b_3, u_1 u_2 u_3 e^{i\alpha_2 b_2}) ,
\]

\[
\tilde{g}_1 \tilde{g}_2 \tilde{g}_3 = (a_1 + a_2 + a_3, b_1 + b_2 + b_3, u_1 u_2 u_3 e^{i\alpha_3 b_3})
\]

are clearly the same: \( \tilde{g}_1 \tilde{g}_2 \tilde{g}_3 = \tilde{g}_1 (\tilde{g}_2 \tilde{g}_3) \).

(b) Cyclic subgroup needs elements

\( g_1, g_2, \ldots, g_N = e \), such that

\[
\tilde{g}_N = \tilde{g}_1^N . \quad \text{But} \quad \tilde{g}_1^N = e = (0, 0, 1) \quad \text{is only possible for} \quad \tilde{g}_1 = (0, 0, u_1) \quad \text{with} \quad u_1^N = 1. \quad \text{Therefore, we have}
\]

\[
\tilde{g}_N = (0, 0, e^{i\frac{2\pi}{N} N})
\]

for the choice \( \tilde{g}_1 = (0, 0, e^{i\frac{2\pi}{N}}) \). Other choices are possible, but they just amount to a permutation of the \( \tilde{g}_N \) and do not give other subgroups.

(c) Need \( e^{i0_1 b_2} = e^{i0_2 b_1} \), that is: \( 0_1 b_2 = 0_2 b_1 \), must be an integer multiple of \( 2\pi \).

(d) According to (c), we have \( b_0 = \frac{0_1}{0_2} \), then

\[
\tilde{g}_{ij} = a^i b^k = (0_1 a_1, k b_0, 1)
\]

with \( j, k = 0, 1, \pm 2, \pm 3, \ldots \), must all be in the subgroup, and we do not need any other element of \( G \), so that these \( \tilde{g}_{ij} \) make up that Abelian subgroup.
(a) Recall

\[ F(t) = \int_0^\infty dt \, e^{-\alpha t} \cdot f(t), \]

\[ -\frac{d}{ds} F(s) = \int_0^\infty dt \, e^{-\alpha t} \cdot f(t), \]

\[ -f(0) + s F(s) = \int_0^\infty dt \, e^{-\alpha t} \cdot \frac{d}{ds} f(t), \]

Then

\[ \int_0^\infty dt \, e^{-\alpha t} + \frac{d^2}{dt^2} f(t) = f(0) - \frac{d}{ds} (\alpha^2 F(s)) \]

\[ = f(0) - 2 \alpha F(s) - \alpha^2 \frac{dF}{ds}. \]

Therefore,

\[ (f(0) - 2 \alpha F(s) - \alpha^2 \frac{dF}{ds}) + 2(-f(0) + s F(0)) - \frac{dF}{ds} = G(s), \]

\[ \text{or} \quad \frac{dF}{ds} = \frac{f(0) + G(s)}{1 + \alpha^2}. \]

(ii) This implies

\[ t f(t) = f(0) \sin t + \int_0^t G(t') \sin(t-t') \]

so that

\[ f(t) = \frac{\sin t}{t} \left( f(0) + \int_0^t G(t') \cos t' \right) \]

\[ - \cos t \int_0^t G(t') \sin t'. \]

Check: The differential equation can be written as

\[ (\frac{d^2}{dt^2} + 1) (tf(t)) = g(t), \]

so that

\[ \text{sin} t \text{ and } \text{cos } t \text{ are obviously the homogeneous solution.} \]
Write answers on this side of the paper only.

\[ B(b)(t) f(t) = \frac{\sin t}{t} \sin t - \frac{\cot t}{t} (1 - \cot t) = \frac{1 - \cot^2 t}{t}. \]

(ii) \[ f(t) = \frac{\sin t}{t} \left[ -2 + 2 (1 - \cot t)^3 \right] - \frac{\cot t}{t} \left[ 2 \cot t \right]^3 \]

\[ = - \frac{\sin (2t)}{t}. \]

(c) Use \( \frac{1}{t} = \int_0^\infty ds \ e^{-st} \), to get

\[ \int_0^\infty dt \ \frac{J_0(at) - J_0(bt)}{t} \]

\[ = \int_0^\infty ds \int_0^\infty dt \ e^{-st} \left[ J_0(at) - J_0(bt) \right] \]

\[ = \int_0^\infty ds \left( \frac{1}{\sqrt{s^2 + a^2}} - \frac{1}{\sqrt{s^2 + b^2}} \right) \]

\[ = \left[ \text{Arsinh} \left( \frac{a}{s} \right) - \text{Arsinh} \left( \frac{b}{s} \right) \right] \big|_s^\infty \]

\[ = \lim_{s \to \infty} \left[ \ln \frac{s + \sqrt{s^2 + a^2}}{b} - \ln \frac{s + \sqrt{s^2 + b^2}}{a} \right] \]

\[ = \ln \frac{b}{a}. \]
We have

\[
\phi(z) = \sum_{k=0}^{\infty} \left[ \frac{1}{3} \left( \frac{3}{z} \right)^k + \frac{3}{z^2} \left( -\frac{1}{z^2} \right)^k + \frac{1}{z} \left( -\frac{1}{z} \right)^k \right]
\]

\[
= \frac{1}{3} \frac{1}{1-3z/3} + \left( \frac{3}{z^2} + \frac{1}{z} \right) \frac{1}{1+1/z^2}
\]

\[
= \frac{1}{3-z} + \frac{3+z^2}{1+z^2} = \frac{10}{(3-z)(1+z^2)} .
\]

The singularities are simple poles at \( z = 3 \) and \( z = \pm i \), with the residues

- at \( z = 3 \), residue = \(-1\); \\
- at \( z = +i \), residue = \( \frac{3-i}{2i} = \frac{1}{2} - \frac{3}{2} i \); \\
- at \( z = -i \), residue = \( \frac{3+i}{-2i} = \frac{1}{2} + \frac{3}{2} i \).

(b) \( \phi(z) = \sum_{k=0}^{\infty} \left[ \frac{1}{3} \left( \frac{3}{3} \right)^k + (3+z) \left( -z^2 \right)^k \right] \)

\[
= \sum_{n=0}^{\infty} b_n z^n , \text{ for } |z| < 1,
\]

with \( b_n = \left( \frac{1}{3} \right)^{n+1} + 3 (-1)^{n/2} \) for \( n \) even

and \( b_n = \left( \frac{1}{3} \right)^{n+1} + (-1)^{(n-1)/2} \) for \( n \) odd.

(c) This path encircles the pole at \( z = 3 \) counterclockwise, so that the integral has the value

\[
2\pi i \cdot (-1) = -2\pi i .
\]

(d) This path encircles the pole at \( z = +i \).
4. (Continued) Counterclockwise around the pole at $z = -i$ counterclockwise, so that the integral has the value

$$2\pi i \times \left( \frac{1}{2} - \frac{3i}{2} \right) - 2\pi i \times \left( \frac{1}{2} + \frac{3i}{2} \right)$$

$$= 6\pi.$$