Problem 1 (20=6+8+6 points)
A harmonic oscillator (natural frequency \( \omega \), ladder operators \( A, A^\dagger \), unperturbed Hamilton operator \( H_0 = \hbar \omega A^\dagger A \) is exposed to a time-independent perturbation that is specified by \( H_1 = \hbar (\Omega A^\dagger + \Omega^* A) \). At the initial time \( t = 0 \), the oscillator is in the ground state of \( H_0 \).

(a) How large is the energy spread \( \delta H \) for \( H = H_0 + H_1 \)?

(b) What is the scattering operator \( S(T) = e^{iH_0T/\hbar} e^{-iHT/\hbar} \) to first order in \( \Omega \)?

(c) What is the probability, to lowest order in \( \Omega \), for finding the oscillator in the 1st, 2nd, 3rd, \ldots excited state of \( H_0 \) after time \( T \) has elapsed?

Problem 2 (25=5+8+5+7 points)
The state of a particle of mass \( M \) is described by the wave function

\[
\psi(\vec{r}, t) = C \frac{x + iy}{r} e^{-r/a} e^{-i\omega t},
\]

where \( C > 0 \), \( a > 0 \), and \( \omega > 0 \).

(a) Determine the normalization constant \( C \).

(b) Find the probability density \( \rho(\vec{r}, t) \) and the probability current density \( \vec{j}(\vec{r}, t) \).

(c) Verify that they obey the continuity equation.

(d) Show that

\[
\frac{d}{dt} \int (d\vec{r}) \rho(\vec{r}, t) = \int (d\vec{r}) \vec{j}(\vec{r}, t)
\]

holds, either by a general argument or by an explicit calculation for the \( \rho(\vec{r}, t) \) and \( \vec{j}(\vec{r}, t) \) of part (b).

Hint: Remember that \( \vec{s} \cdot \nabla \vec{r} = \vec{s} \) for any 3-vector \( \vec{s} \).
**Problem 3** (25=12+8+5 points)
A particle of mass \( M \) is scattered by the Gaussian potential

\[
V(\vec{r}) = V_0 e^{-\frac{1}{2} (r/a)^2} \quad \text{with} \quad a > 0 \quad \text{and} \quad V_0 = \frac{(\hbar/a)^2}{2M}.
\]

Apply the first-order Born approximation and determine

(a) the differential cross section \( \frac{d\sigma}{d\Omega(\theta)} \) in terms of \( a \) and \( q = 2k \sin \frac{\theta}{2} \);

(b) the total cross section \( \sigma \) in terms of \( a \) and \( E/V_0 \) with \( E = \frac{(\hbar k)^2}{2M} \);

(c) the dominating \( E \) dependence for \( E \ll V_0 \) and \( E \gg V_0 \).

Hint: Remember that \( d\theta \sin \theta = \frac{1}{k^2} dq \).

**Problem 4** (30=5+10+10+5 points)
Two-level atom; probability amplitudes for states 1 and 2 are \( \psi_1 \) and \( \psi_2 \), respectively.

The Schrödinger equation for \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \) is

\[
\frac{i\hbar}{\partial t} \psi(t) = \mathcal{H}(t) \psi(t)
\]

with \( \mathcal{H}(t) = \hbar \omega \begin{pmatrix} \cos(2\phi(t)) & \sin(2\phi(t)) \\ \sin(2\phi(t)) & -\cos(2\phi(t)) \end{pmatrix} \) where \( \phi(t) = \pi t/T \) with \( T > 0 \).

(a) Find the eigencolumns \( \psi_{\pm}(t) \) of \( \mathcal{H}(t) \) to the eigenvalues \( \pm \hbar \omega \).

(b) Write \( \psi(t) = \alpha(t) \psi_+(t) + \beta(t) \psi_-(t) \) and find the 2 \times 2 matrix \( \mathcal{M} \) in

\[
\frac{\partial}{\partial t} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = i \mathcal{M} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}.
\]

[Check: If you get it right, \( \mathcal{M} \) does not depend on \( t \).]

(c) By solving this equation, find \( \psi(T) \) for \( \psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

(d) What is the dominating \( T \) dependence of the probability \( |\psi_2(T)|^2 \) for \( \omega T \ll 1 \)? And what is it for \( \omega T \gg 1 \)?