The Frauchiger-Renner No-go Theorem is Quantum Fake News

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Abstract

In this report, we look into Daniela Frauchiger and Renato Renner's "Quantum theory cannot consistently describe the use of itself" [1] and show how three refutations can be used to argue that the AR might be rejected without foregoing any of their proposed assumptions (Q), (S) and (C), if one analyzes particular premises of the reasoning (denoted as $\mathbb{Q}_i$). This is to because of hidden affirmations of contradictory premises regarding whether certain qubits have been measured or not. We then further emphasize the absurdity of these fallacious premises by reproducing them in the context of Greenberger-Horne-Zeilinger experiments and also by reiterating them infinitely.

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Notation & Conventions

Agents	Quantum System	Pointer
$\bar{F} \rightarrow$ Alice	$\left\|\mathsf{heads}\right\rangle_R \rightarrow \left\|h\right\rangle_A, \left\|\mathsf{tails}\right\rangle_R \rightarrow \left\|t\right\rangle_A$	$\left\|\bar{\mathsf{h}}\right\rangle_{\bar{L}} \rightarrow \left\|H\right\rangle_{P_A}, \left\|\bar{\mathsf{t}}\right\rangle_{\bar{L}} \rightarrow \left\|T\right\rangle_{P_A}$
$F \rightarrow$ Bob	$\left\|\downarrow\right\rangle_S \rightarrow \left\|\downarrow\right\rangle_B, \left\|\uparrow\right\rangle_S \rightarrow \left\|\uparrow\right\rangle_B$	$\left\|-\frac{1}{2}\right\rangle_L \rightarrow \left\|\Downarrow\right\rangle_{P_B}, \left\|+\frac{1}{2}\right\rangle_L \rightarrow \left\|\Uparrow\right\rangle_{P_B}$
$\bar{W} \rightarrow$ Yvette	$\left\|\bar{\mathsf{ok}}\right\rangle_{\bar{L}} \rightarrow \left\|-Y\right\rangle, \left\|\mathsf{ok}\right\rangle_{\bar{L}} \rightarrow \left\|+Y\right\rangle$	-
$W \rightarrow$ Wigner	$\left\|\bar{\mathsf{ok}}\right\rangle_L \rightarrow \left\|+W\right\rangle, \left\|\mathsf{ok}\right\rangle_L \rightarrow \left\|-W\right\rangle$	-

Table 1: Mapping of Frauchiger & Renner's notation to this paper's notation

FR Notation	This Paper's Notation
$\mathbb{Q}_1 \rightarrow \left\langle\mathsf{init}\right\| \pi^{n:10}_{(\bar{w},z)=(\bar{\mathsf{ok}},-\frac{1}{2})} \left\|\mathsf{init}\right\rangle_R$	$\mathbb{Q}_1 \rightarrow \left\langle\Psi_i\right\| \pi^{n:10}_{-Y,\downarrow} \left\|\Psi\right\rangle_i$
$\pi^{n:10}_{(\bar{w},z)=(\bar{\mathsf{ok}},-\frac{1}{2})} = (\hat{U}^{00\rightarrow 10}_{R\rightarrow\bar{L}S})^\dagger \left\|\bar{\mathsf{ok}}\right\rangle_{\bar{L}} \left\|\downarrow\right\rangle_S \left\langle\downarrow\right\|_S \left\langle\bar{\mathsf{ok}}\right\|_{\bar{L}} (\hat{U}^{00\rightarrow 10}_{R\rightarrow\bar{L}S})$	$\pi^{n:00}_{-Y,\downarrow} = [\hat{\mathbf{U}}_1^\dagger \left\|-Y\right\rangle \left\|\downarrow\right\rangle_B \left\langle\downarrow\right\|_B \left\langle -Y\right\| \hat{\mathbf{U}}_1]$
$\left\|\mathsf{init}\right\rangle_R = \sqrt{\frac{2}{3}} \left\|\mathsf{tails}\right\rangle_R + \sqrt{\frac{1}{3}} \left\|\mathsf{heads}\right\rangle_R$	$\left\|\Psi\right\rangle_i = \sqrt{\frac{2}{3}} \left\|t\right\rangle_A + \sqrt{\frac{1}{3}} \left\|h\right\rangle_A$
$\hat{U}^{00\rightarrow 10}_{R\rightarrow\bar{L}S} = \begin{cases} \left\|\mathsf{heads}\right\rangle_R \rightarrow \left\|\bar{\mathsf{h}}\right\rangle_{\bar{L}} \otimes \left\|\downarrow\right\rangle_S \\ \left\|\mathsf{tails}\right\rangle_R \rightarrow \left\|\bar{\mathsf{t}}\right\rangle_{\bar{L}} \otimes \left\|\rightarrow\right\rangle_S \end{cases}$	$\hat{\mathbf{U}}_1^\dagger = \begin{cases} \left\|h\right\rangle_A \rightarrow \left\|H\right\rangle_{P_A} \otimes \left\|\downarrow\right\rangle_B \\ \left\|t\right\rangle_A \rightarrow \left\|T\right\rangle_{P_A} \otimes \left\|\rightarrow\right\rangle_B \end{cases}$

Table 2: Regarding $\mathbb{Q}_1$

FR Notation	This Paper's Notation
$\pi^{n:10}_{z=-\frac{1}{2}} = \left\|\downarrow\right\rangle \left\langle\downarrow\right\|_S$	$\pi^{n:10}_{\Downarrow} = \left\|\downarrow\right\rangle \left\langle\downarrow\right\|_B$
$\pi^{n:10}_{z=+\frac{1}{2}} = \left\|\uparrow\right\rangle \left\langle\uparrow\right\|_S$	$\pi^{n:10}_{\Uparrow} = \left\|\uparrow\right\rangle \left\langle\uparrow\right\|_B$

Table 3: Regarding $\mathbb{Q}_2$

FR Notation	This Paper's Notation
$\mathbb{Q}_3 \rightarrow \langle \rightarrow \vert \, \pi^{n:10}_{w=\mathsf{ok}} \, \vert \rightarrow \rangle_S$	$\mathbb{Q}_3 \rightarrow \langle \rightarrow \vert \, \pi^{n:10}_{-W} \, \vert \rightarrow \rangle_B$
$\pi^{n:10}_{w=\mathsf{ok}} = (\hat{U}^{10\rightarrow 20}_{S\rightarrow L})^\dagger \, \vert \mathsf{ok} \rangle_L \, \langle \mathsf{ok} \vert_L \, (\hat{U}^{10\rightarrow 20}_{S\rightarrow L})$	$\pi^{n:10}_{-W} = [\hat{\mathbf{U}}^\dagger_3 \, \vert -W \rangle \, \langle -W \vert \, \hat{\mathbf{U}}_3]$
$\hat{U}^{10\rightarrow 20}_{S\rightarrow L} = \begin{cases} \vert \downarrow \rangle_S \rightarrow \vert -\frac{1}{2} \rangle_S \\ \vert \uparrow \rangle_S \rightarrow \vert +\frac{1}{2} \rangle_S \end{cases}$	$\hat{\mathbf{U}}^\dagger_3 = \begin{cases} \vert \downarrow \rangle_B \rightarrow \vert \Downarrow \rangle_{P_B} \\ \vert \uparrow \rangle_B \rightarrow \vert \Uparrow \rangle_{P_B} \end{cases}$

Table 4: Regarding $\mathbb{Q}_3$

Used in Deductive Arguments[1]

Let A and B be propositions or a set of propositions

Denotation	Description
$\Leftrightarrow$	`Material Biconditional` s.t. A if and only if B.
$\rightarrow$	`Material Conditional` s.t. $A \rightarrow B \Leftrightarrow$ If A then B.
$\wedge$	`Logical Conjunction` s.t. $A \wedge B \Leftrightarrow A$ and B.
$\vee$	`Inclusive Disjunction` s.t. $A \vee B \Leftrightarrow A$ (inclusive) or B.
$\oplus$	`Exclusive Disjunction` s.t. $A \oplus B \Leftrightarrow$ Either A or B.
$\neg$	`Negation` s.t. $\neg A \Leftrightarrow$ Not A.

De Morgan's Laws

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$$

Note then that:

$$\begin{aligned} \neg(A \wedge B \wedge C) &\Leftrightarrow \neg((A \wedge B) \wedge C) \\ &\Leftrightarrow \neg(A \wedge B) \vee \neg C \\ &\Leftrightarrow \neg A \vee \neg B \vee \neg C \end{aligned} \tag{1}$$

Conventions & Acronyms

p_i : i-th premise

c_i : i-th conclusion

c_f : Final conclusion

FR : Frauchiger & Renner

FRTE : FR Thought Experiment

AR : Alternative Reasoning

CR : Calculative Reasoning

$\times$: Declined premise / conclusion

[1] Extracted from [2] [3] [4].

Valid Deductive Inferences

`Modus Ponens` **MP**

p_1	$A \rightarrow B$	If A, then B.
p_2	A	A.
c_f	$\therefore B$	Therefore, B.

`Modus Tollens` **MT**

p_1	$A \rightarrow B$	If A, then B.
p_2	$\neg B$	Not B.
c_f	$\therefore \neg B$	Therefore, not A.

`Hypothetical Syllogism` **HS**

p_1	$A \rightarrow B$	If A, then B.
p_2	$B \rightarrow C$	If B, then C.
c_f	$\therefore A \rightarrow C$	Therefore, if A, then C.

`Implies as Disjunct`

$$(A \rightarrow B) \Leftrightarrow (\neg A \vee B)$$

`Contrapositive` **CP**

$$(A \rightarrow B) \Leftrightarrow (\neg B \rightarrow \neg A)$$

`A Compound Rule` **CX**

Consider:

p_1	$(A \wedge B) \rightarrow C$	
p_2	$\neg C$	
c_1	$\neg(A \wedge B)$	**MT**
$-$	$\neg A \vee \neg B$	De Morgan's Laws
$-$	$A \rightarrow \neg B$	Implies as Disjunct
p_3	A	
c_f	$\neg B$	**MP**

Therefore:

p_1	$(A \wedge B) \rightarrow C$	
p_2	$\neg C$	
p_3	A	
c_f	$\neg B$	**CX**

1 Introduction

1.1 Who Cares? - Thought Experiments, Fake News & Personal Motivation

Quantum theory gets a fair bit of attention in both popular science circles and everyday media. Though it is properly seen as remarkably successful and empirically robust, its coverage in these non-academic circles has often been fixated on its apparent "spookiness". This is particularly obvious in the kind of emphasis given to various thought experiments associated with quantum theory, such as the famous (or, rather, infamous) Schrödinger's cat thought experiment.

This might be why many physicists often find themselves disillusioned if not downright cynical about discussions concerning thought experiments. The late professor Hawking once quipped that "[w]hen I hear of Schrödinger's cat, I reach for my gun." [5] This sort of outlook probably has much to do with how thought experiments have a tendency to generate what might be called scientific fake news, perpetuating misreadings and the misuse of quantum theory.

As someone who has been pursuing my degree in physics in hopes to later further my studies in the philosophy of science, I hope to be able to put thought experiments in their proper place. To neither fall back on the extreme of scientific fake news nor the extreme of unwarranted cynicism. One thinks of how the discussion of Maxwell's Demon helped do away with bad definitions of entropy, and developed a deeper understanding of thermodynamics alongside important clarifications from the philosophy of science [6] [7] [8] [9] [10]. Therefore, it seems best to neither aggrandize nor wholly ignore thought experiments that arise. But to instead examine the claims of thought experiments before evaluating these claims accurately and consequently dismantle the fake news that it might generate. This is the broad goal and motivation of my paper.

That said, we can now turn to the case at hand.

1.2 The Frauchiger & Renner Thought Experiment

Instead of looking at older thought experiments (which have sufficient academic coverage already), I have set my sights on a novel one by Daniela Frauchiger and Renato Renner [1][2] which have been the subject of quite a lot of academic controversy (or "buzz"). This is not very surprising, given that the paper is audaciously titled: "Quantum Theory Cannot Consistently Describe the Use of Itself". The controversy and critiques of the FRTE have been so spirited that it has become an academic meme of sorts, passed around generating discussion amongst the interested and appalled alike. This discussion, however, is associated with not only some very ambitious claims about quantum foundations, but also pretty "far-out" claims about topics pertinent in philosophy (such as the correspondence theory of truth, modal logic and philosophy of science).

[2] I will, hereby, refer to this thought experiment as FRTE, while referring to Frauchiger and Renner as FR.

Let me list some of these "far-out" claims about FRTE (or, at least, aspects of it):

- Shows that agents can know they exist "in two different worlds" [11]
- "Spiritually leans towards Many Worlds" view of quantum theory [12]
- Can help adjudicate between interpretations of quantum theory [1] [13][3] [16]
- Shows quantum theory is not universally valid, by virtue of it being inconsistent when describing agents that are using it. Agents using the same theories and rules come to contradictory conclusion. [1]
- Shows classical logic (namely, modal logic) is inapplicable in quantum settings. [14] [15]
- Shows that observer-independence can be ruled out if we assume free choice and locality. [13]
- Shows that scientific results, under quantum theory, may not be objectively true (it is agent-relative). [14]

Now, it is one thing to be convinced of any one of these statements (that is, via other arguments and notions). But it is a whole other thing to say these statements are somehow verified or vindicated by *this* thought experiment. It is the latter that is the real issue here. From these statements we see that it is quite clear that the FRTE can potentially grow into a "garbage fire" of scientific fake news unless understood well.[4]

[3]The state used in this paper differs from the one employed in the FR paper, but nevertheless takes after the FRTE's set-up and arguments in many ways. For instance, the use of "Wigners" and "friends", and "Wigners" that measure in the diagonal basis etc.

[4]One might only look to the various comment sections on the internet discussing this paper to find responses and personal insights of all kinds, trying to take apart what this thought experiment must mean (take for instance, Scott Aaronson's blog. [17]). This is not unexpected, since the thought experiment has been seen as a kind of unholy matrimony of the Wigner's Friend [18] and Hardy "paradoxes" [19].

1.3 The Approach Taken

With that said, I list here is a step-by-step forecast of how I will approach this fascinating (and appalling) thought experiment:

1. **Introduction:** As I am currently doing, I first introduce my motivations and lay out the goals of the paper.
2. **Set-Up:** I will first describe the set-up of the thought experiment from which FR's argument emerges.
3. **The Frauchiger & Renner Argument:** I will then explain what FR's argument is in detail before giving my sensing of where the critical question (or angle of most interest) ultimately lies.

How I arrived at this "critical question" is really found in section 6. However, because this concerns an analysis of the more logical calculus of the FR paper, I have chosen to classify this as a *review* of the paper that can be read after all the physics-concerned sections. Those who want a clearer and more logically systematic understanding of my evaluation can consider going to section 6 prior to section 4.

4. **Three Refutations:** I will then launch a total of three possible ways to show that FR's conclusions are unwarranted. In particular, I will argue that one cannot arrive at their no-go theorem, given the real fallacy in the AR, which I will identify via a number of different angles.
   (a) **Hardy's Problem & Equivocation:** Through comparisons with the Hardy Problem and how it shows FR's AR conceals fallacy of equivocation.
   (b) **Isometries & Contradictions:** Through an examination of FR's isometries and how these reveal a contradiction within the reasoning of the experimenters.
   (c) **Memory Encoding & Information:** How FR's argument still breaks down even if we grant them a means out of the contradiction discussed in the previous section on Isometries.
5. **Extension & Exploration:** Finally, I will explore some extensions of the FRTE as well. This is done to illustrate "just how wrong" the fallacy is. I do this in two ways: (1) by recontextualizing the fallacy into Greenberger-Horne-Zeilinger experiments and (2) by reiterating it infinitely, showing we produce absurdities by virtue of this fallacy, wholly apart from any violation of (Q), (C) and (S).
6. **Review & Deductive Logic Analysis** As mentioned, this portion actually serves as the backdrop logically prior to the "Three Refutations" section. This is where I will simply delineate how I arrived at the approach I employed in the previous section. Namely, via a syllogistic analysis of FR's AR and seeing if these really map over that well to their (Q), (C) and (S) assumptions, as their no-go theorem would require.
7. **Conclusions:** I will then summarize all my findings and conclude.

With this course of discussion in mind, let's get down to it.

2 The Set-Up

2.1 Initializations

The set up consists of two labs of "Wigner's friend" type experimenters Alice and Bob. Each lab consists of (1) a quantum object to be measured, (2) an experimenter and (3) their respective measurement devices. Each of these "friends" have their own "Wigner", who measures their lab as a whole.[5] Let's call these two agents Yvette and Wigner. Yvette measures Alice's Lab. Wigner measures Bob's Lab (see figure 2).

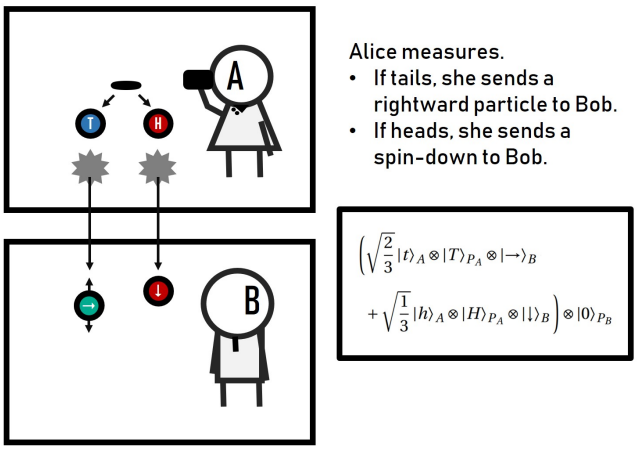


Figure 1: Alice measures her quantum biased coin and sends a qubit to Bob's lab

Hence, two pairs of experimenters and a total of four agents.

Now, an important aside: the mathematical description of the experiment I will be laying out will be that of one combined wavefunction, utilizing tensor products and null states. This aimed to help the experimental process easier to follow. In the original FR paper, however, they opted to use isometries as transformations to describe every step. Both approaches, I find, are virtually the same and the difference has no impact on the emergence of FR's no-go theorem proposal. But, even if one insists that this small difference in approach will somehow totally miss FR's argument, rest assured the refutation found in "Isometries & Contradictions" (Section 4.3) will employ their mathematical approach, and give us the same conclusions.

[5] Of course, how this could be even physically conceivable in an actual, physical lab is completely up for grabs and debate. But as one will see, the point of the FRTE is not whether it is physically feasible, but what it tells us about quantum theory as a scientific theory.

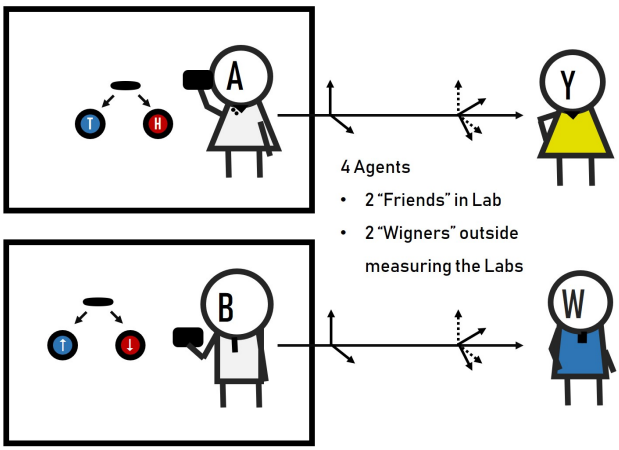


Figure 2: All four agents.

2.2 Alice & Bob

Now, we begin with Alice having a quantum-biased coin of the following weightage:

$$|\Psi\rangle_i = \sqrt{\frac{2}{3}}\,|t\rangle_A + \sqrt{\frac{1}{3}}\,|h\rangle_A \tag{2}$$

Hence, mapping out the rest of the set-up (alongside the pointers) we get:

$$|\psi\rangle_{FR,1} = \left(\sqrt{\frac{2}{3}}\,|t\rangle_A + \sqrt{\frac{1}{3}}\,|h\rangle_A\right) \otimes |\cdot\rangle_{P_A} \otimes |\cdot\rangle_B \otimes |\cdot\rangle_{P_B} \tag{3}$$

So here the subscripts P_A, B, P_B refer to Alice's pointer (that is, her measurement device), a spin-particle for Bob's lab (that he will receive later on) and Bob's pointer respectively. Naturally, at this stage, these all remain in the null state $|\cdot\rangle$. This initial state in (3), before Alice measures her quantum biased coin corresponds to FR's timestamp "before n:00". These timestamps may not seem very important at this juncture but will be of note in the refutations section.

Now we move on to the next stage. Alice measures her quantum biased coin and then, depending on what she had measured, sends a spin-particle qubit to Bob. If she finds the coin in $|t\rangle$, she prepares a qubit of state $|{\rightarrow}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + |{\downarrow}\rangle)$ to Bob's lab. On the other hand, if she finds the coin in $|h\rangle$, she sends a qubit in the state $|{\downarrow}\rangle$ to Bob. Pretty straightfoward so far. All in all, we get the following description:

$$|\psi\rangle_{FR,2} = \left(\sqrt{\frac{2}{3}}\,|t\rangle_A \otimes |T\rangle_{P_A} \otimes |{\rightarrow}\rangle_B + \sqrt{\frac{1}{3}}\,|h\rangle_A \otimes |H\rangle_{P_A} \otimes |{\downarrow}\rangle_B\right) \otimes |\cdot\rangle_{P_B} \tag{4}$$

As it should be, the only thing that remains in the null state here is Bob's pointer. Since he has yet to measure the qubit he had received from Alice. The above state (4) corresponds to FR's interval "from n:00 to n:10".

Finally, Bob measures his qubit. And so the state, might be described as such:

$$|\Psi\rangle_{FR} = \sqrt{\frac{1}{3}}\Bigg(|t\rangle_A \otimes |T\rangle_{P_A} \otimes |\uparrow\rangle_B \otimes |\Uparrow\rangle_{P_A} + \Bigg(|h\rangle_A \otimes |H\rangle_{P_A} + |t\rangle_A \otimes |T\rangle_{P_A} \Bigg) \otimes |\downarrow\rangle_B \otimes |\Downarrow\rangle_{P_B} \Bigg) \quad (5)$$

This corresponds to FR's time interval "from n:10 to n:20". This completed wavefunction $|\Psi\rangle_{FR}$, will be of primary importance throughout this paper. Also note that $|\Psi\rangle_{FR}$ can be identified as a "Hardy state":

$$|\Psi\rangle_{HP} = \sqrt{\frac{1}{3}}\Big(|+z\rangle_a \otimes |+z\rangle_b + |+z\rangle_a \otimes |-z\rangle_b + |-z\rangle_a \otimes |+z\rangle_b \Big) \quad (6)$$

Or, more compactly,

$$|\Psi\rangle_{HP} = \sqrt{\frac{1}{3}}\Big(|+z,+z\rangle + |+z,-z\rangle + |-z,+z\rangle \Big) \quad (7)$$

This will be significant particularly in section 4.2. With (5) in mind, we then move on the interactions of Wigner and Yvette who are now going to measure the labs of their respective "friends".

2.3 Yvette & Wigner

Now, Yvette measures Alice's lab and Wigner measures Bob's lab. However, unlike in the original Wigner's friend thought experiment, here these two agents measure in the diagonal or "Hadamard" basis of the labs. What this means is simply that Yvette measures using the following basis:

$$|\pm Y\rangle = \frac{|h\rangle_A \otimes |H\rangle_{P_A} \pm |t\rangle_A \otimes |T\rangle_{P_A}}{\sqrt{2}} \quad (8)$$

While Wigner uses this basis:

$$|\pm W\rangle = \frac{|\uparrow\rangle_B \otimes |\Uparrow\rangle_{P_B} \pm |\downarrow\rangle_B \otimes |\Downarrow\rangle_{P_B}}{\sqrt{2}} \quad (9)$$

This is essentially the set-up for the thought experiment. By itself, it does not really "say anything" at all. We must thus move on to what FR's argument, which will show us why such outlandish statements have been generated out of this thought experiment.

A full pictorial outline of the experiment is included below:

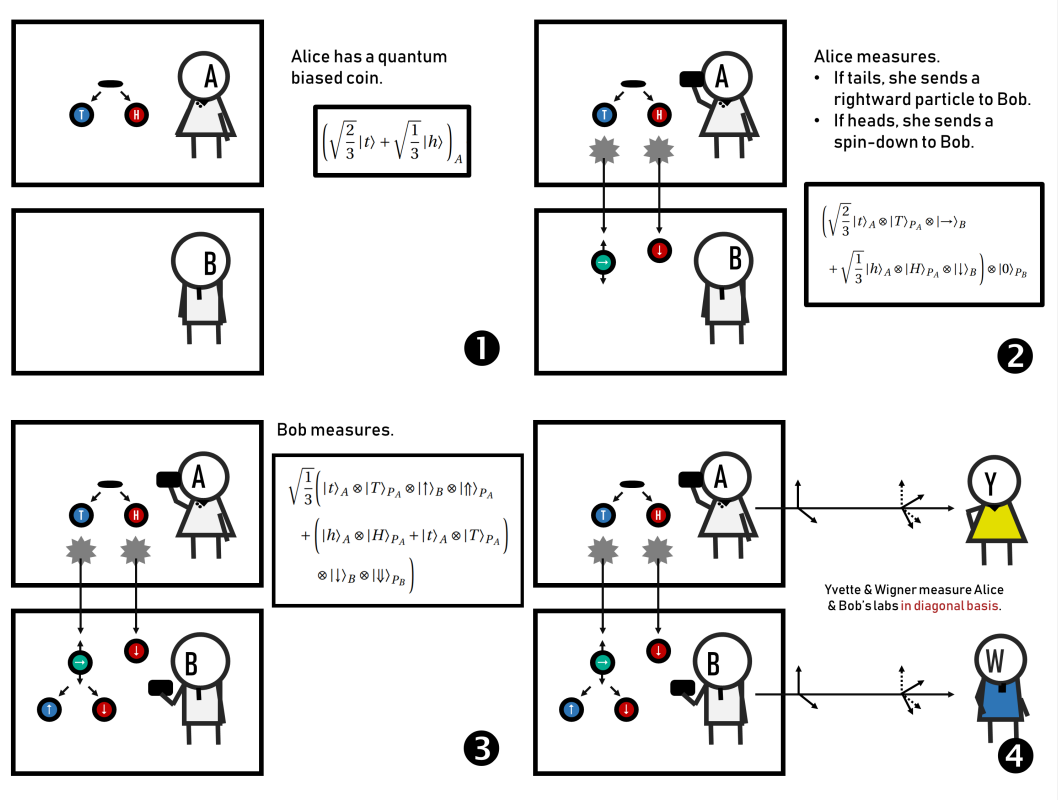


Figure 3: Pictorial outline of the thought experiment's procedure.

3 The Frauchiger & Renner Argument

3.1 Two Contradictory Results

Out of this set-up, FR produce two lines of reasoning that give two conclusions about the probability of Wigner measuring $|-W\rangle$, given that Yvette measured $|-Y\rangle$ (i.e $P(-Y,-W)$). I will call these two lines of reasoning "calculative reasoning" and "alternative reasoning" respectively.

3.1.1 Calculative Reasoning (CR)

Here, we simply follow the sort of reasoning that is characteristic to the quantum paradigm. One way to find $P(-Y,-W)$ for $|\Psi\rangle_{FR}$ would be to acquire the square of $\langle -Y| \otimes \langle -W| \, |\Psi\rangle_{FR}$.

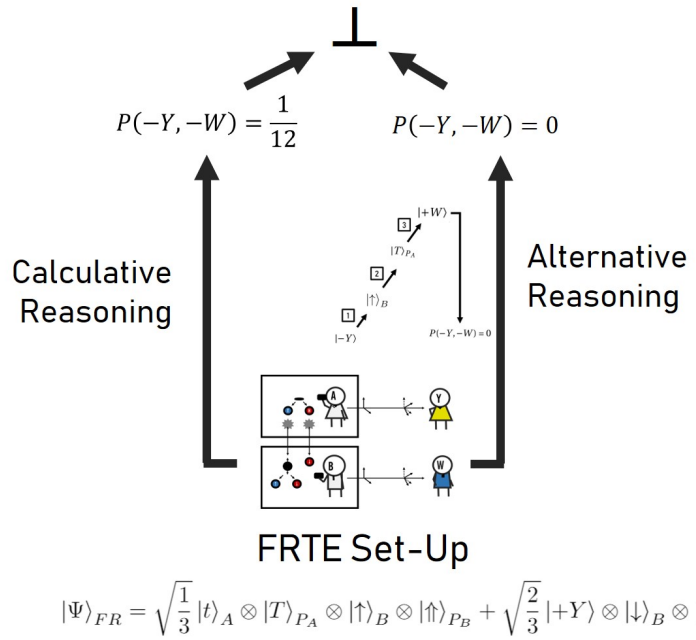


Figure 4: Diagram depicting the contradiction result emergent from the lines of reasoning.

Hence, we find,

$$
\begin{aligned}
&\langle -Y|\otimes\langle -W|\,|\Psi\rangle_{FR}\\
&\quad=\langle -Y|\otimes\langle -W|\sqrt{\frac{1}{3}}\Bigg(|t\rangle_A\otimes|T\rangle_{P_A}\otimes|\uparrow\rangle_B\otimes|\Uparrow\rangle_{P_A}+\Bigg(|h\rangle_A\otimes|H\rangle_{P_A}+|t\rangle_A\otimes|T\rangle_{P_A}\Bigg)\otimes|\downarrow\rangle_B\otimes|\Downarrow\rangle_{P_B}\Bigg)\\
&\quad=\langle -Y|\otimes\langle -W|\sqrt{\frac{1}{3}}\Bigg(|t\rangle_A\otimes|T\rangle_{P_A}\otimes|\uparrow\rangle_B\otimes|\Uparrow\rangle_{P_A}+\sqrt{2}|+Y\rangle\otimes|\downarrow\rangle_B\otimes|\Downarrow\rangle_{P_B}\Bigg)\\
&\quad=\langle -Y|\otimes\langle -W|\sqrt{\frac{1}{3}}\Bigg(|t\rangle_A\otimes|T\rangle_{P_A}\otimes|\uparrow\rangle_B\otimes|\Uparrow\rangle_{P_A}\Bigg)\\
&\quad=\langle -Y|\otimes\langle -W|\sqrt{\frac{1}{3}}\Bigg(\sqrt{\frac{1}{2}}\Big(|+Y\rangle-|-Y\rangle\Big)\otimes\sqrt{\frac{1}{2}}\Big(|+W\rangle+|-W\rangle\Big)\Bigg)\\
&\quad=-\sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}=-\sqrt{\frac{1}{12}}
\end{aligned}
\tag{10}
$$

Consequently, via the "calculative reasoning" (i.e everyday quantum theory and hereby referred to as CR), we get a **non-zero** probability for $P(-Y,-W)$, namely:

$$
P(-Y,-W)=\frac{1}{12} \tag{11}
$$

3.1.2 Alternative Reasoning (AR)

But FR propose that the agents could employ another sort of quantum theoretical reasoning that I have christened the uninspired label "alternative reasoning" (hereby referred to as AR). Skipping a whole lot of details[6], the main gist of this AR can be laid out like this:

First off, consider how $|\Psi\rangle_{FR}$ in (5) can be rewritten as such (noting (8)):

$$|\Psi\rangle_{FR} = \sqrt{\frac{1}{3}}\,|t\rangle_A \otimes |T\rangle_{P_A} \otimes |\uparrow\rangle_B \otimes |\Uparrow\rangle_{P_B} + \sqrt{\frac{2}{3}}\,|{+}Y\rangle \otimes |\downarrow\rangle_B \otimes |\Downarrow\rangle_{P_B} \tag{12}$$

- Now, let's say that Yvette does her measurement of Alice's lab and finds that it is in $|{-}Y\rangle$.
- By inspection, she properly notes that this means the second term in (12) will go to zero. Hence, she concludes Alice's pointer and coin must be in the state $|t\rangle_A \otimes |T\rangle_{P_A}$.
- Yvette then considers how Alice being in $|t\rangle_A \otimes |T\rangle_{P_A}$ implies that Bob's lab is in $|{\rightarrow}\rangle_A$. How so? Simply via (4).
- Since Bob's lab is in $|{\rightarrow}\rangle_A$ (which is identical to $\frac{1}{\sqrt{2}}(|\uparrow\rangle_B + |\downarrow\rangle_B)$, this necessarily means that $\frac{1}{\sqrt{2}}(|\uparrow\rangle_B - |\downarrow\rangle_B)$ as an orthogonal state is not possible for Bob's lab.
- This, in turn, implies that $|{-}W\rangle$, which corresponds to $\frac{1}{\sqrt{2}}(|\uparrow\rangle_B - |\downarrow\rangle_B)$ as in (9), is not a possible state to describe Bob's lab. Hence, Wigner will never measure $|{-}W\rangle$.
- Hence, together, if Yvette observes lab in $|{-}Y\rangle$, then $|{-}W\rangle$ is not a possible state.

Therefore, through this reasoning,

$$P(-Y, -W) = 0 \tag{13}$$

Box 1: *AR, Provisional Rundown*

The **zero probability** result of (13) produced by this dubious AR stands in contradiction to the **non-zero probability** found from the CR (11).

[6] See the more expanded and intentional analysis in section 6

3.1.3 What's with the Alternative Reasoning?

Now, the obvious question is to ask why this alternative path was raised at all, especially since the familiar CR is clearly the one that actually works. Why does FR put forward the AR when it is so clearly mistaken?

Hence, to clear up potential misreadings, it should be said that FR does *not* propose the FRTE and the AR to show that our regular quantum calculations are all wrong. Rather they propose them to argue that, by virtue of this set-up and the emergent contradiction, the AR *has to be mistaken* and thus, FR argues, one of three assumptions that underlie this reasoning must be rejected. This is this is what consists of their **no-go theorem**, and it is to this we will now turn.

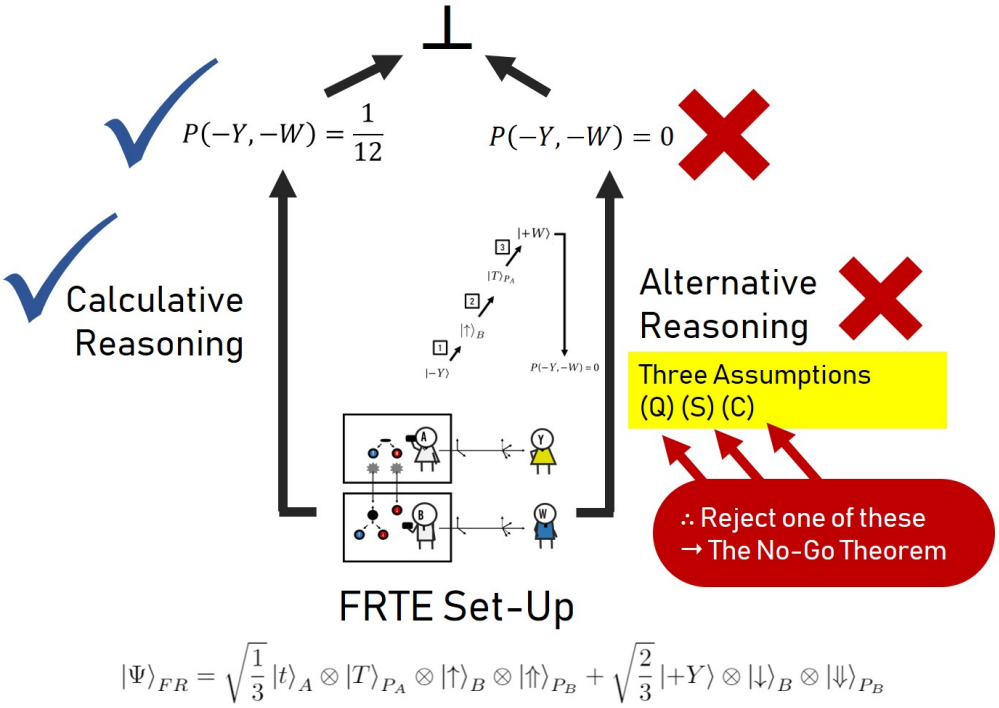


Figure 5: A pictorial map of the FR argument.

3.2 The FR No-go Theorem

FR's no-go theorem states that

1. "[a]ny theory that satisfies assumptions (Q), (C), and (S) yields contradictory statements when applied to the [FRTE]"[7]
2. and that though it "does not tell us which of these three assumptions is wrong"[8],
3. nevertheless "it implies that any specific interpretation of quantum theory, when applied to the *Gedankenexperiment*, will necessarily conflict with at least one of them."[9]

Hence, the no-go theorem forces us to reject some combination of these given assumptions (Q), (C) and (S)[10]. Now, what are these (Q), (C) and (S) anyhow? Extended answers to this have been given by FR[11], but brief summaries are sufficient at this stage:

- **(Q)uantum Born Rule**
  - (Q) refers to the use of quantum born rule by each of the experimenters.
  - To **reject** this (i.e ¬(Q)) is to say that quantum theory is foundationally mistaken.
- **(C)onsistency**
  - (C) refers to how agents using the exact same theoretical framework should always yield the same conclusions.
  - To **reject** this (i.e ¬(C)) would be to say that agents using the same rules and ideas in quantum theory can give contradictory scientific results. (FR leans toward this option. This is what they mean by their title "quantum theory cannot consistently describe the use of itself".)
- **(S)ingle Outcome**
  - (S) refers to how one cannot simultaneously affirm a scientific result and its negation.
  - To **reject** this (i.e ¬(S)) is to essentially reject the law of non-contradiction.

Box 2: *FR's (Q), (C) and (S) Assumptions*

[7] [1], Page. 8
[8] [1], Page. 2
[9] [1], Page. 2
[10] In logical notation, ¬(Q) ∨ ¬(C) ∨ ¬(S).
[11] [1], Pages. 4, 7, 8. Also consider section 6.2 for a more detailed take on the literature's description of the three assumptions

Now we see how FR's no-go theorem presents us with an uneasy trichotomy of sorts. In light of the contradiction between the CR and AR, and given how letting go of any of these options is quite difficult, we thus apprehend the pointed edge of FR's arguments which have generated so much discussion.

3.3 The Overall Argument of the FR Paper & Critical Question

The FR argument for the no-go theorem can thus be summarized roughly as follows also see 5:

1. Two possible lines of reasoning emerge from the set-up for (12)
   (a) "Calculative Reasoning" (CR)
   (b) "Alternative Reasoning" (AR)
2. CR is the typical way we do calculations in quantum mechanics, and is undisputed.
3. CR will result in the conclusion that $P(-Y, -W) = 1/12$
4. AR consists of three main assumptions, designated (Q), (S), (C).
5. AR will result in the conclusion that $P(-Y, -W) = 0$
6. These results contradict.
7. Since we know the CR is not in dispute, the AR must be false.
8. Hence (Q), (S), (C) cannot all be true. (Thus, FR's "No-go Theorem")

Box 3: *The Overall Argument, Delineated*[12]

See how it is clearly a misconception to think showing or claiming that the AR is wrong somehow debunks FR's paper. This is to simply state step 7, which FR affirm. Their claim is **not** that "the AR is the correct way of thinking, therefore quantum theory is wrong". Rather, **it is** "the AR is obviously the wrong way of thinking. So why is that?" and then arguing for their "no-go theorem" from there.

[12] For a more formal version of this consider Deductive Argument 7 in section 6.1.

In this way, we see how the main avenue to contest the no-go theorem and to avoid the trichotomy is to contest step *4* instead. What this means is that the critical question is really this:

The Critical Question

Is the AR's main assumption only (Q), (C) and (S)? In other words, is the rejection of (Q), (C) and (S) the ***only*** way to show the AR is false?

If there are other ways to show that the AR is unsound, then (Q), (S) and (C) can be preserved within every quantum theoretical framework.

Box 4: *The Critical Question to answer in the following sections*

4 Three Refutations

4.1 Approach to the Critical Question

To answer the critical question it seems then we must do the following:

1. Breakdown the AR to its respective premises.
2. Check if these respective premises map over to (Q), (S) and (C) only.
3. If they do not, identify what are the extra assumptions in addition to (Q), (S) and (C).
4. Investigate if these extra assumptions could explain the emergent contradiction in the FR argument.

Now, section 6 "Review & Deductive Logic Analysis" deals with the first three parts and can be read at the end of this paper since as they are not that heavy on physics and concern a syllogistic analysis of FR's intricate argument. Having done all that I have found that there are three premises in the AR that are of most interest with regard to the critical question. These are designated $\mathbb{Q}_1$, $\mathbb{Q}_2$ and $\mathbb{Q}_3$ and correspond to the following facts:

	Designation	In Propositional Calculus	In English	Probability
p_2	$\mathbb{Q}_1$	T0 $\rightarrow$ T1	If Yvette measures $\|-Y\rangle$ at n:21 then she is certain Bob knows he measured $\|\uparrow\rangle_B$ at n:11	$P(-Y,\uparrow)=1$
p_4	$\mathbb{Q}_2$	T2 $\rightarrow$ T3	If Bob measures $\|\uparrow\rangle_B$ at n:11 then he is certain Alice knows she measured $\|t\rangle_A$ at n:01	$P(\uparrow,t)=1$
p_6	$\mathbb{Q}_3$	T4 $\rightarrow$ T5	If Alice measures $\|t\rangle_A$ at n:01 then she is certain that "Wigner will observe $\|+W\rangle$ at n:31"	$P(t,-W)=0$

Table 5: The three $\mathbb{Q}_i$ premises and their place in the AR tabulated.

It seems while the rest of the premises in the AR correspond well to (C), (S) and some trivial matters, these three $\mathbb{Q}_i$ steps do not map over to (Q) as well as FR would require.

With this in mind,[13] I lay out my thesis:

FR's no-go theorem is mistaken. This is because there is a contradiction in the premises of the AR as they present it in their paper (most particularly in **Table 3**). Hence, one does not need to reject either (Q), (C) or (S) to show the AR is unsound. The AR is unsound simply by virtue of a contradiction hidden in its argumentation.

And the contradiction is this: the AR affirms $\mathbb{Q}_3$ while also affirming $\mathbb{Q}_1$ and $\mathbb{Q}_2$. But these premises are require mutually exclusive assumptions about whether quantum systems are measured or not.[a]

Finally, this contradiction between the premises can be shown in at least three ways.

[a] Where $\mathbb{Q}_1$, $\mathbb{Q}_2$ and $\mathbb{Q}_3$ is defined in the table above and also in Deductive Argument 8

Box 5: *My Thesis against the FR Argument & No-Go Theorem*

[13] I give a full scale analysis in section 6 for this very conclusion

4.2 Hardy's Problem & Equivocation

One way to refute the FR argument is to make a comparison of their set-up with the Hardy Problem [19]. This will naturally require some background in the physics and notation associated to Bell's inequality and Bell tests. [20] [21]

4.2.1 Brief Review of the Original Hardy Problem

We begin by seeing how the FRTE wavefunction corresponds to the so-called Hardy State. Copying (7),

$$|\Psi\rangle_{HP} = \sqrt{\frac{1}{3}}\Big(|+z,+z\rangle + |+z,-z\rangle + |-z,+z\rangle \Big) \tag{14}$$

The first component corresponds to a system designated "a" while the second component corresponds to a system "b". Hence, we can develop some notation: Take "$a_z = +$" to mean that "a measurement of the first system 'a' gave $|+z\rangle$". Likewise, "$b_x = -$" means that "a measurement of the second system 'b' gave $|-x\rangle$". Where the x-basis is the diagonal basis of the z-basis.

Further still, now let $P(+-|zx) \Leftrightarrow P(a_z = +, b_x = -)$. Or, in plain english, let $P(+-|zx)$ be the conditional probability that given "$a_z = +$" we also have "$b_x = -$". And so on for all $P(\pm\pm|^z/_x{}^z/_x)$. Now Hardy's paradox further states that through both quantum theory calculations and experimental findings we will acquire certain statistics which we can loosely call "rules":

$$P(+-|zx) = 0 \tag{15}$$

$$P(--|zz) = 0 \tag{16}$$

$$P(-+|zx) = 0 \tag{17}$$

Box 6: *Hardy Problem "Rules"*

The calculations are straightforward. Consider (15):

$$\begin{aligned}
P(+-|zx) &= \mathsf{Tr}\Big[|\Psi\rangle_{HP} \langle\Psi|_{HP} |+z,-x\rangle \langle+z,-x| \Big] \\
&= \mathsf{Tr}\Big[\sqrt{\frac{1}{3}} |\Psi\rangle_{HP} \Big(\langle+z,+z| + \langle+z,-z| + \langle-z,+z| \Big) |+z,-x\rangle \langle+z,-x| \Big] \\
&= \mathsf{Tr}\Big[\sqrt{\frac{1}{3}} |\Psi\rangle_{HP} \Big(\langle+z|_b + \langle-z|_b \Big) |-x\rangle_b \langle-x|_b \Big] \\
&= \mathsf{Tr}\Big[\sqrt{\frac{2}{3}} |\Psi\rangle_{HP} \langle+x|_b |-x\rangle_b \langle-x|_b \Big] = 0
\end{aligned}$$

Similarly, for (16) and (17):

$$P(--|zz) = \mathsf{Tr}\Big[\,|\Psi\rangle_{HP}\langle\Psi|_{HP}\,|-z,-z\rangle\langle -z,-z|\,\Big] = 0$$
$$P(-+|xz) = \mathsf{Tr}\Big[\,|\Psi\rangle_{HP}\langle\Psi|_{HP}\,|-x,+z\rangle\langle -x,+z|\,\Big] = 0$$

And so one of the interesting things about the Hardy state is that **if we assume local hidden variables** we will get the proposition "$P(--|xx) = 0$" deductively via these rules. That is, if the systems or particles that correspond to "a" and "b" are unable to instantaneously communicate to each other what basis in which they are being measured ($\because$ local hidden variables), then they will need to "come up with certain *strategies*" (speaking analogously for the hidden variables that the systems possess with respect to each other) to govern how they behave in order to fulfill the rules. These strategies will manifest in a script of five statements:

Rules		a_z	a_x	b_z	b_x	Implication
		+	+	-	+	
$P(+-\|zx) = 0$		+	-	-	+	
$P(--\|zz) = 0$	$\xRightarrow[\text{Form "strategies"}]{\text{Assume LHV}}$	-	+	+	+	$\Rightarrow \quad P(--\|xx) = 0$
$P(-+\|xz) = 0$		-	+	+	-	
		+	+	+	+	

For the Hardy State (as in the original Hardy problem), assuming **local hidden variables** can give us the conclusion that $P(--|xx) = 0$.

Box 7: *Hardy Problem "Rules"*

Now, from this table we realize that $P(--|xx) = 0$ is zero also (given the assumptions made), since none of the above strategies allow a scenario where both $a_x = -$ and $b_x = -$. So this is why the Hardy state is quite an interesting case: by assuming local hidden variables we can show that it yields $P(--|xx) = 0$. Yet, the fact remains that if we use the proper CR in quantum theory (or conduct an experiment on two entangled particles jointly describable by the Hardy state), $P(--|xx) \neq 0$. Consider:

$$\begin{aligned}
P(--|xx) &= \mathsf{Tr}\Big[\,|\Psi\rangle_{HP}\langle\Psi|_{HP}\,|-x,-x\rangle\langle -x,-x|\,\Big] \\
&= \frac{1}{2}\mathsf{Tr}\Big[\sqrt{\frac{1}{3}}\,|\Psi\rangle_{HP}\Big(\langle +z,+z| + \langle +z,-z| + \langle -z,+z|\Big)\Big(|+z\rangle_A - |-z\rangle_A\Big)\otimes\Big(|+z\rangle_B - |-z\rangle_B\Big)\langle -x,-x|\,\Big] \\
&= \frac{1}{2}\mathsf{Tr}\Big[\sqrt{\frac{1}{3}}\,|\Psi\rangle_{HP}\Big(\langle +z|_B + \langle -z|_B - \langle +z|_B\Big)\Big(|+z\rangle_B - |-z\rangle_B\Big)\langle -x,-x|\,\Big] = -\frac{1}{2}\mathsf{Tr}\Big[\sqrt{\frac{1}{3}}\,|\Psi\rangle_{HP}\langle -x,-x|\,\Big] \quad (18) \\
&= -\frac{1}{12}\mathsf{Tr}\Big[\Big(|+z,+z\rangle + |+z,-z\rangle + |-z,+z\rangle\Big)\Big(\langle +z,+z| - \langle +z,-z| - \langle -z,+z| + \langle -z,-z|\Big)\Big] \\
&= -\frac{1}{12}\cdot(1-1-1) = \frac{1}{12}
\end{aligned}$$

Herein, then, lies the "problem" in "Hardy's problem". But really the main thing is just that Hardy's state can be used to illustrate a violation of locality. I delineate the argument as a logical calculus here below[14].

–	Propositional Calculus	Relevant Section / Equation
p_1	$\|\Psi_{HP}\rangle$	See (14).
p_2	$\|\Psi_{HP}\rangle \rightarrow$ "Rules"	See Box 6.
c_1	$\therefore$ "Rules"	From p_1, p_2 **MP**
p_3	("Rules" $\wedge$ Locality) $\rightarrow P(--\|xx) = 0$	See Box 7.
p_4	$P(--\|xx) \neq 0$	Empirically & via CR.
c_2	$\therefore \neg$(Locality)	From p_3, p_4, c_1 **MT,CX**
c_f	$\therefore$ Nonlocality	From c_2

Deductive Argument 1: Nonlocality Argument for the Hardy State

4.2.2 The Alternative Reasoning as an Argument in Bell Notation

Having discussed the original Hardy Problem we see how it mirrors the FR argument in many ways. Both arguments produce a conundrum surrounding a dubious zero result for $P(--|xx)$, but via different assumptions and means. While the AR in Hardy Problem uses the assumption of local hidden variables, the AR in the FR argument uses a different sort of contrivance. Consider first how the FR state in (5) can be mapped over nicely to the notation we have been using[15]:

$\pm$	a_z			a_x			b_z			b_x		
	HP	$\Rightarrow$	FR	HP	$\Rightarrow$	FR	HP	$\Rightarrow$	FR	HP	$\Rightarrow$	FR
$+$	$\|+z\rangle_a$	$\Rightarrow$	$\|tT\rangle_A$	$\|+z\rangle_b$	$\Rightarrow$	$\|\downarrow\Downarrow\rangle_B$	$\|+x\rangle_a$	$\Rightarrow$	$\|+Y\rangle$	$\|+x\rangle_b$	$\Rightarrow$	$\|+W\rangle$
$-$	$\|-z\rangle_a$	$\Rightarrow$	$\|hH\rangle_A$	$\|-z\rangle_b$	$\Rightarrow$	$\|\uparrow\Uparrow\rangle_B$	$\|-x\rangle_a$	$\Rightarrow$	$\|-Y\rangle$	$\|-x\rangle_b$	$\Rightarrow$	$\|-W\rangle$

Table 6: Conversion between notations of the original hardy problem and the FR problem.

Let us also rewrite (5) from the FRTE into this:

$$|\Psi\rangle_{FR} = \sqrt{\frac{1}{3}}\Big(|tT,\downarrow\Downarrow\rangle + |tT,\uparrow\Uparrow\rangle + |hH,\downarrow\Downarrow\rangle \Big) \tag{19}$$

[14] This might be of some interest later on when we enter talk of creating a Greenberger-Horne-Zeilinger (GHZ) version of the FRTE.

[15] Where $|tT\rangle_A \rightarrow |t\rangle_A \otimes |T\rangle_{P_A}$ and so on

Notice then how the "rules" of the original Hardy problem (15), (16) and (17) are naturally also satisfied by the FR state in (5). For completeness, consider how (19) can be found for (15):

$$\begin{aligned}
P(+-|zx) &= \mathsf{Tr}\Big[\,|\Psi\rangle_{FR}\langle\Psi|_{FR}\,|tT,-W\rangle\langle tT,-W|\,\Big] \\
&= \mathsf{Tr}\Big[\sqrt{\frac{1}{3}}\,|\Psi\rangle_{FR}\Big(\,|tT,{\downarrow\Downarrow}\rangle + |tT,{\uparrow\Uparrow}\rangle + |hH,{\downarrow\Downarrow}\rangle\Big)\Big(\,|tT,-W\rangle\langle tT,-W|\,\Big)\Big] \\
&= \mathsf{Tr}\Big[\sqrt{\frac{1}{3}}\,|\Psi\rangle_{FR}\Big(\,\langle{\uparrow\Uparrow}|_b + \langle{\downarrow\Downarrow}|_b\Big)\Big(\,|-W\rangle\langle -W|\,\Big)\Big] \\
&= \mathsf{Tr}\Big[\sqrt{\frac{2}{3}}\,|\Psi\rangle_{FR}\langle +W|-W\rangle\langle -W|\,\Big] = 0
\end{aligned}$$

Similarly, for (16) and (17):

$$\begin{aligned}
P(--|zz) &= \mathsf{Tr}\Big[\,|\Psi\rangle_{FR}\langle\Psi|_{FR}\,|hH,{\uparrow\Uparrow}\rangle\langle hH,{\uparrow\Uparrow}|\,\Big] = 0 \\
P(-+|xz) &= \mathsf{Tr}\Big[\,|\Psi\rangle_{FR}\langle\Psi|_{FR}\,|-Y,{\downarrow\Downarrow}\rangle\langle -Y,{\downarrow\Downarrow}|\,\Big] = 0
\end{aligned}$$

FR's AR can thus be described in Bell test notation:

–	Propositional Calculus	Relevant Section / Equation
p_1	$(a_x = -) \rightarrow (b_z = -)$	$P(-+\|xz)$, $\mathbb{Q}_1$
p_2	$(b_z = -) \rightarrow (a_z = +)$	$P(--\|zz)$, $\mathbb{Q}_2$
p_3	$(a_z = +) \rightarrow (b_x = +)$	$P(+-\|zx)$, $\mathbb{Q}_3$
c_1	$(a_x = -) \rightarrow (b_x = +)$	From p_1, p_2, p_3 HS,HS
c_2	$\therefore P(--\|xx) = 0$	$(b_x = +) \rightarrow \neg(b_x = -)$, Mathematically.
c_f	$\therefore P(-Y,-W) = 0$	Yvette & Wigner correspond to the x-basis.

Deductive Argument 2: FR's Alternative Reasoning in Bell Notation (To be Resolved)

4.2.3 Analysis & Evaluation

Now that we have framed the AR in such a way, we can begin to see how the FR argument and no-go theorem is wrong by showing that a logical fallacy is what results in the zero probability for $P(-Y,-W)$ and none of the three assumptions (Q), (S) or (C) need to be discarded.

Firstly, consider how there is something quite suspicious about p_1 and p_2 from the above deductive argument. How can they be held as true at the same time? Especially when $a_x \Rightarrow \neg a_z$? That is, that in quantum contexts if a system "a" is measured in the x-basis then it definitely *cannot* be simultaneously taken as having been measured in the z-basis. Yet if we are able to string together p_1 and p_2 (as required for c_1), then it demands that $a_x \Rightarrow a_z$. Which is absurd. Either a given quantum system is measured in x or that system is measured in z, and in a strictly exclusive way. You cannot have both simultaneously.

So what has gone wrong with this argument? Each statement (p_1, p_2 and p_3) by itself seems sound enough. But yet together, they entail physical impossibilities and contradictions. With some inspection, the issue seems to be that the syllogism's deductive structure is invalid, by virtue of a logical fallacy of "equivocation" [22]. p_1's antecedent clause $\left(b_z = -\right)_{p_1}$ cannot be mapped over to p_2's consequent clause $\left(b_z = -\right)_{p_2}$ and hence c_1 *cannot* be deduced by the preceding premises p_i. This is because $\left(b_z = -\right)_{p_1}$ is acquired *after* measuring system "a" while $\left(b_z = 1\right)_{p_2}$ is acquired *before* measuring system "a". The same thing applies the stringing between p_2 and p_3, $\left(a_z = +\right)_{p_2}$ is not the same as $\left(a_z = +\right)_{p_3}$ since they assume different things about whether system "b" has been measured or not.

Thus, c_1 cannot be concluded and, importantly, without rejecting any (Q), (C) or (S). Which simply implies that FR's no-go theorem is wrong.

–	Propositional Calculus	Relevant Section / Equation
p_1	$(a_x = -) \rightarrow \left(b_z = -\right)_{p_1}$	$P(-+\|xz)$, $\mathbb{Q}_1$, noting $\left(b_z = -\right)_{p_1} \neq \left(b_z = -\right)_{p_2}$
p_2	$\left(b_z = -\right)_{p_2} \rightarrow \left(a_z = +\right)_{p_2}$	$P(--\|zz)$, $\mathbb{Q}_2$, noting $\left(a_z = +\right)_{p_2} \neq \left(a_z = +\right)_{p_3}$
p_3	$\left(a_z = +\right)_{p_3} \rightarrow (b_x = +)$	$P(+-\|zx)$, $\mathbb{Q}_3$
×	$(a_x = -) \rightarrow (b_x = +)$ ×	From p_1, p_2, p_3 ×
×	$\therefore P(--\|xx) = 0$ ×	$(b_x = +) \rightarrow \neg(b_x = -)$, Mathematically.
×	$\therefore P(-Y, -W) = 0$ ×	Yvette & Wigner correspond to the diagonal basis.

Deductive Argument 3: Deductive Argument 2 with Equivocation Accounted For.

In this way, we see how the AR can be rejected without rejecting (Q), (S) or (C), since it contains a logical fallacy of equivocation.

4.2.4 Further Thoughts

Now we see how, quite interestingly, that for the Hardy state we can use two sorts of means to arrive at the same empirically false conclusion. We can (1) assume local hidden variables or (2) employ a strange reasoning involving a fallacy of equivocation. This comparison should mean that it is possible to feign a non-violation of bell-inequality by FR's contrived reasoning alongside the surfaced manner of equivocation fallacy. I will show how this is the case more explicitly with the GHZ-FRTE hybrid thought experiment in subsection 5.1.

4.3 Isometries & Contradiction

It is still possible that an advocate of FR's theorem charges that this fails to respect the argument's terminology and steps. Hence, here we take a closer look into the steps of the argument which correspond to what has been identified as the problematic premises of Deductive Argument 8. These would be the $\mathbb{Q}_i$ premises. Copying table 5:

	Designation	In Propositional Calculus	In English	Probability
p_2	$\mathbb{Q}_1$	T0 $\rightarrow$ T1	If Yvette measures $\|-Y\rangle$ at n:21 then she is certain Bob knows he measured $\|\uparrow\rangle_B$ at n:11	$P(-Y,\uparrow)=1$
p_4	$\mathbb{Q}_2$	T2 $\rightarrow$ T3	If Bob measures $\|\uparrow\rangle_B$ at n:11 then he is certain Alice knows she measured $\|t\rangle_A$ at n:01	$P(\uparrow,t)=1$
p_6	$\mathbb{Q}_3$	T4 $\rightarrow$ T5	If Alice measures $\|t\rangle_A$ at n:01 then she is certain that "Wigner will observe $\|+W\rangle$ at n:31"	$P(t,-W)=0$

As dictated by the no-go theorem, denying any of these $\mathbb{Q}_i$ requires us to reject the Born rule itself. We will now perform the calculations proposed by FR and see whether this surfaces some flaws in their argument.

4.3.1 Analysis of $\mathbb{Q}_1$

For FR, $\mathbb{Q}_1$ consists of Yvette reasoning via Born rule that the joint probability of her measuring $|-Y\rangle$ and Bob measuring $|\uparrow\rangle_B$ is 1. That is, $P(-Y,\uparrow)=1$. The way they establish this is through the equation labelled (6) in their paper. Also relevant would be **Table 2** of their paper.[16]

$$
\begin{aligned}
P(-Y,\uparrow) &= \langle\Psi|_i\,\pi^{n:00}_{(-Y,\uparrow)}\,|\Psi\rangle_i = \langle\Psi|_i\left(\mathbf{1}-\pi^{n:00}_{(-Y,\downarrow)}\right)|\Psi\rangle_i = \langle\Psi|_i\,|\Psi\rangle_i + \langle\Psi|_i\,\pi^{n:00}_{(-Y,\downarrow)}\,|\Psi\rangle_i \\
&= 1 + \langle\Psi|_i\,\hat{\mathbf{U}}_1^\dagger\,|-Y\rangle\,|\downarrow\rangle_B\,\langle\downarrow|_B\,\langle -Y|\,\hat{\mathbf{U}}_1\,|\Psi\rangle_i \\
\because \hat{\mathbf{U}}_1\,|\Psi\rangle_i &= \hat{\mathbf{U}}_1\left(\sqrt{\frac{2}{3}}\,|t\rangle_A + \sqrt{\frac{1}{3}}\,|h\rangle_A\right) = \sqrt{\frac{1}{3}}\,|H\rangle_{P_A}\otimes|\downarrow\rangle_B + \sqrt{\frac{2}{3}}\,|T\rangle_{P_A}\otimes|\rightarrow\rangle_B \\
&= \sqrt{\frac{1}{3}}\Big(\,|H\rangle_{P_A}+|T\rangle_{P_A}\,\Big)\otimes|\downarrow\rangle_B + \sqrt{\frac{1}{3}}\,|T\rangle_{P_A}\otimes|\uparrow\rangle_B \\
&= \sqrt{\frac{2}{3}}\,|+Y\rangle\otimes|\downarrow\rangle_B + \sqrt{\frac{1}{3}}\,|T\rangle_{P_A}\otimes|\uparrow\rangle_B \\
\therefore P(-Y,\uparrow) &= 1-\langle\Psi|_i\,\hat{\mathbf{U}}_1^\dagger\,|-Y\rangle\,|\downarrow\rangle_B\,\langle\downarrow|_B\,\langle -Y|\left(\sqrt{\frac{2}{3}}\,|+Y\rangle\otimes|\downarrow\rangle_B + \sqrt{\frac{1}{3}}\,|T\rangle_{P_A}\otimes|\uparrow\rangle_B\right) \\
&= 1-\langle\Psi|_i\,\hat{\mathbf{U}}_1^\dagger\,|-Y\rangle\,|\downarrow\rangle_B\,\langle\downarrow|_B\cdot 0 = 1
\end{aligned}
\tag{20}
$$

[16] Take note of how my notation maps over to the FR Paper's in tables 2, 3, 4.

$\therefore$ "If Yvette measures $|-Y\rangle$ at n:21 then she is certain Bob knows he measured $|\uparrow\rangle_B$ at n:11"

4.3.2 Analysis of $\mathbb{Q}_2$

As for $\mathbb{Q}_2$, FR has Bob reasoning via Born rule that the joint probability of him measuring $|\uparrow\rangle_B$ and Alice measuring $|t\rangle_A$ is 1 (i.e $P(\uparrow, t) = 1$). Thus, making him certain that she had measured $|t\rangle_A$ rather than $|h\rangle_A$.

$$\begin{aligned} P(\uparrow, t) &= \langle\uparrow|_B \, \pi_{\Uparrow}^{n:10} \, |\uparrow\rangle_B \\ &= \langle\uparrow|\uparrow\rangle_B \, \langle\uparrow|\uparrow\rangle_B = 1 \end{aligned} \tag{21}$$

$\therefore$ "If Bob measures $|\uparrow\rangle_B$ at n:11 then he is certain Alice knows she measured $|t\rangle_A$ at n:01". This is probably also combined with the FRTE's given premise that Alice would have sent a $|\downarrow\rangle_B$ if she had measured $|h\rangle_A$ and so the only possible $|\uparrow\rangle_B$ component must come from the $|\rightarrow\rangle_B$ that Alice sends if she gets $|t\rangle_A$.

4.3.3 Analysis of $\mathbb{Q}_3$

Now, $\mathbb{Q}_3$ on the other hand consists of Alice reasoning via Born rule that the joint probability of her measuring $|t\rangle_A$ and Wigner measuring $|-W\rangle$ is 0. That is, $P(t, -W) = 0$.

$$\begin{aligned} P(t, -W) &= \langle\rightarrow|_B \, \pi_{(-W)}^{n:10} \, |\rightarrow\rangle_B \\ &= \langle\rightarrow|_B \, \hat{\mathbf{U}}_3^\dagger \, |-W\rangle \, \langle -W| \, \hat{\mathbf{U}}_3 \, |\rightarrow\rangle_B \\ \hat{\mathbf{U}}_3 \, |\rightarrow\rangle_B &= \hat{\mathbf{U}}_3 \sqrt{\frac{1}{2}} \Big(\, |\downarrow\rangle_B + |\uparrow\rangle_B \, \Big) \\ &= \sqrt{\frac{1}{2}} \Big(\, |\Downarrow\rangle_{P_B} + |\Uparrow\rangle_{P_B} \, \Big) \\ &= |+W\rangle \\ \therefore P(t, -W) &= \langle +W|-W\rangle \, \langle -W|+W\rangle = 0 \end{aligned} \tag{22}$$

$\therefore$ "If Alice measures $|t\rangle_A$ at n:01 then she is certain that 'Wigner will observe $|+W\rangle$ at n:31'"

4.3.4 Evaluation: Surfacing a Glaring Contradiction

Now that we have done the calculations for the various isometries and Heisenberg operators that FR claims to correspond to the agents using the quantum born rule well in order to achieve full certainty about their colleagues, we can finally move on to evaluating if these steps are in fact airtight.

And indeed, with some inspection a glaring contradiction emerges. In the argument that leads to the false conclusion, we have to combine two contradictory statements about Bob's qubit. Namely, FR's proposed quantum born calculations affirm two mutually exclusive conditions. Consider how $\mathbb{Q}_2$ requires the conditional clause: "If Bob measures $|\uparrow\rangle_B$ at

n:11" while the calculation that has been ascribed to $\mathbb{Q}_3$ requires that Bob's qubit remain in $|\rightarrow\rangle_B$ as seen in (22)'s very first step. So which is it? Is Bob's qubit in $|\uparrow\rangle_B$ as required in $\mathbb{Q}_2$ or is it $|\rightarrow\rangle_B$ as required by $\mathbb{Q}_3$? Has Bob done the measurement or not?

Whatever the case may be, we see that the argument breaks down. Without rejecting (Q) (S) or (C), the AR proposed by FR is shown to be wrong wholly because of a contradiction in the reasoning process of the agents. To pull through with their argumentation they must assume two conditions that are mutually exclusive. How is it, then, a surprise that their reasoning produces a contradiction when it is a contradiction that has been input into the syntax in the first place? Thusly, no one needs to reject (Q) (S) or (C) since the argument in many ways rejects itself.

From this evaluation of the FR argument, we see there are two ways to reject the FR no-go theorem.

1. **CONTRADICTION:** Instead of rejecting (Q), (C) and (S), we simply state that the AR of FR has a contradiction:
   - It requires the agents to hold two mutually exclusive statements about the set-up Namely:
     (a) That Bob's qubit **is not measured** and remains in $|\rightarrow\rangle_B$ as required by $\mathbb{Q}_3$ (as implicitly assumed by FR's step of $P(t,-W) = \langle\rightarrow|_B \pi^{n:10}_{(-W)} |\rightarrow\rangle_B$)
     (b) That Bob's qubit **is measured** and is now found in $|\uparrow\rangle_B$ as required by $\mathbb{Q}_2$.
   - These cannot both be true, hence the AR is wrong without rejecting (Q), (C) or (S).
2. **DENY PREMISE $\mathbb{Q}_3$:** Instead of rejecting of (Q), (C) and (S) we simply deny $\mathbb{Q}_3$ and say that though "Alice measures $|t\rangle_A$ at n:01" she *cannot* be certain that 'Wigner will observe $|+W\rangle$ at n:31' since "she does not know if the wavefunction at hand will be in anyway changed" between n:01 and n:31.

Either way the no-go theorem is avoided.

However, let us be even more modest to FR argument's proponent. Perhaps there is still a move that he could employ in order to preserve the no-go theorem. It such a move that we will now address and evaluate.

4.4 Memory Encoding & Information

As suggested, perhaps an amendment could be made about the FRTE so that the contradiction surfaced might be avoided. Since the contradiction lies in whether Bob's qubit has or has not been measured perhaps we can make it such that Bob's qubit has two component, one that is measured by Wigner (for $\mathbb{Q}_3$) while another component is measured by Bob (and thus the subject of $\mathbb{Q}_2$).

The Amendment (To Avoid Contradiction between $\mathbb{Q}_2$ & $\mathbb{Q}_3$)

$$\frac{|\uparrow\rangle_B + |\downarrow\rangle_B}{\sqrt{2}} \xRightarrow{\text{amend}} \frac{|\uparrow_W\uparrow_B\rangle + |\downarrow_W\downarrow_B\rangle}{\sqrt{2}} \tag{23}$$

However, this obviously means that certain things will change about the use of Born's rule. Significantly $\mathbb{Q}_3$ must now possess a new component to address the qubit that remains in Bob's lab that is not going to be measured by Wigner.

$$\pi^{n:10}_{(-W)} \xRightarrow{\text{amend}} \pi^{n:10}_{(-W)} \otimes \mathbf{1}_B$$

Also,

$$|\pm W\rangle = \frac{|\Uparrow_W\rangle \pm |\Downarrow_W\rangle}{\sqrt{2}}$$

How then does the calculation for $P(t, -W)$ change? Does the joint probability remain as 0? The answer to that is no. Let us redo the calculation from (22) with the amendments in place.

The probability $P(t,-W)$ can be expressed as such.

$$P(t,-W) = \mathsf{Tr}\Big[\pi^{n:10}_{(-W)} \otimes \mathbf{1}_B \cdot \rho\Big] \tag{24}$$

The density matrix, taking into account the amendments will now be:

$$\begin{aligned}\rho &= \left(\frac{|{\uparrow_W}{\uparrow_B}\rangle + |{\downarrow_W}{\downarrow_B}\rangle}{\sqrt{2}}\right)\left(\frac{\langle{\uparrow_W}{\uparrow_B}| + \langle{\downarrow_W}{\downarrow_B}|}{\sqrt{2}}\right)\\ &= \frac{1}{2}\Big(\,|{\uparrow_W}{\uparrow_B}\rangle\langle{\uparrow_W}{\uparrow_B}| + |{\downarrow_W}{\downarrow_B}\rangle\langle{\downarrow_W}{\downarrow_B}| + |{\downarrow_W}{\downarrow_B}\rangle\langle{\uparrow_W}{\uparrow_B}| + |{\uparrow_W}{\uparrow_B}\rangle\langle{\downarrow_W}{\downarrow_B}|\,\Big)\end{aligned}$$

Finally, the amended operators are as such:

$$\pi^{n:10}_{(-W)} \otimes \mathbf{1}_B = \pi^{n:10}_{(-W)} \otimes \Big(\,|{\uparrow_B}\rangle\langle{\uparrow_B}| + |{\downarrow_B}\rangle\langle{\downarrow_B}|\,\Big)$$

Together, the target probability can be evaluated as such:

$$\begin{aligned}P(t,-W) &= \mathsf{Tr}\Big[\frac{1}{2}\pi^{n:10}_{(-W)} \otimes \Big(\,|{\uparrow_B}\rangle\langle{\uparrow_B}| + |{\downarrow_B}\rangle\langle{\downarrow_B}|\,\Big)\\ &\qquad\cdot\Big(\,|{\uparrow_W}{\uparrow_B}\rangle\langle{\uparrow_W}{\uparrow_B}| + |{\downarrow_W}{\downarrow_B}\rangle\langle{\downarrow_W}{\downarrow_B}| + |{\downarrow_W}{\downarrow_B}\rangle\langle{\uparrow_W}{\uparrow_B}| + |{\uparrow_W}{\uparrow_B}\rangle\langle{\downarrow_W}{\downarrow_B}|\,\Big)\Big]\\ &= \mathsf{Tr}\Big[\frac{1}{2}\pi^{n:10}_{(-W)} \cdot \Big(\,|{\uparrow_W}\rangle\langle{\uparrow_W}| + |{\downarrow_W}\rangle\langle{\downarrow_W}|\,\Big)\Big]\\ &= \mathsf{Tr}\Big[\frac{1}{2}\hat{\mathbf{U}}_3^{\dagger}\,|{-W}\rangle\langle{-W}|\,\hat{\mathbf{U}}_3\Big(\,|{\uparrow_W}\rangle\langle{\uparrow_W}| + |{\downarrow_W}\rangle\langle{\downarrow_W}|\,\Big)\Big]\\ &= \mathsf{Tr}\Big[\frac{1}{2}\,|{-W}\rangle\langle{-W}|\,\Big(\,|{\Uparrow_W}\rangle\langle{\Uparrow_W}| + |{\Downarrow_W}\rangle\langle{\Downarrow_W}|\,\Big)\Big]\\ &= \mathsf{Tr}\Big[\frac{1}{4}\Big(\,|{\Uparrow_W}\rangle - |{\Downarrow_W}\rangle\,\Big)\Big(\,\langle{\Uparrow_W}| - \langle{\Downarrow_W}|\,\Big)\Big(\,|{\Uparrow_W}\rangle\langle{\Uparrow_W}| + |{\Downarrow_W}\rangle\langle{\Downarrow_W}|\,\Big)\Big]\\ &= \mathsf{Tr}\Big[\frac{1}{4}\Big(\,|{\Uparrow_W}\rangle\langle{\Uparrow_W}| + |{\Downarrow_W}\rangle\langle{\Downarrow_W}|\,\Big)\Big]\\ &= \frac{1}{4}\cdot 2\\ \therefore P(t,-W) &= \frac{1}{2} \neq 0\end{aligned}$$

$\therefore$ "If Alice measures $|t\rangle_A$ at n:01 then she is **NOT** certain that 'Wigner will observe $|{+W}\rangle$ at n:31'" $\Rightarrow \mathbb{Q}_3$ is false without denying (Q), (S) or (C) $\Rightarrow$ FR no-go theorem remains unsound.

Thusly, working around the contradiction surfaced in the previous section does not ultimately prove effective avoiding the conclusion that the no-go theorem is in error. The FR argument is still unsound.

4.4.1 Mathematical Resolution

To conclude this section I will look to show how this amendment plays out mathematically, especially after addressing another dubious step found in $\mathbb{Q}_1$. For this, consider the few first steps of (20):

$$\begin{aligned} P(-Y,\uparrow) &= \langle\Psi|_i \, \pi^{n:00}_{(-Y,\uparrow)} \, |\Psi\rangle_i \\ &= \langle\Psi|_i \left(\mathbf{1} - \pi^{n:00}_{(-Y,\downarrow)}\right) |\Psi\rangle_i \end{aligned}$$

These steps[17] assume that

$$\pi^{n:00}_{(-Y,\uparrow)} = \mathbf{1} - \pi^{n:00}_{(-Y,\downarrow)}$$

But is this really the case? Why does Yvette have to reason like this rather than do the calculation as below?

$$\pi^{n:00}_{(-Y,\uparrow)} = \hat{\mathbf{U}}_1^\dagger \, |-Y\rangle \, |\uparrow\rangle_B \, \langle\uparrow|_B \, \langle -Y| \, \hat{\mathbf{U}}_1$$

This only seems as the appropriate counterpart to their definition for $\pi^{n:00}_{(-Y,\downarrow)}$. Hence, recalling that

$$\hat{\mathbf{U}}_1 \, |\Psi\rangle_i = \sqrt{\frac{2}{3}} \, |+Y\rangle \otimes |\downarrow\rangle_B + \sqrt{\frac{1}{3}} \, |T\rangle_{P_A} \otimes |\uparrow\rangle_B$$

We perform the calculation for Yvette's reasoning in $\mathbb{Q}_1$:

$$\begin{aligned} P(-Y,\uparrow) &= \langle\Psi|_i \, \pi^{n:00}_{(-Y,\uparrow)} \, |\Psi\rangle_i \\ &= \langle\Psi|_i \, \hat{\mathbf{U}}_1^\dagger \, |-Y\rangle \, |\uparrow\rangle_B \, \langle\uparrow|_B \, \langle -Y| \, \hat{\mathbf{U}}_1 \, |\Psi\rangle_i \\ &= \frac{1}{3} \cdot \langle T|_{P_A} \, |-Y\rangle\langle -Y| \, |T\rangle_{P_A} \\ &= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \end{aligned}$$

[17] Page 7. **Table 2**, [1].

Interestingly, this gives us the exact same result as the calculative reasoning. Therefore completely undoing the pointed edge of the FR argument:

$\mathbb{Q}_i$	FR Claim	Error	After Amendment	End Result
$\mathbb{Q}_1$	$P(-Y,\uparrow)=1$	$\pi^{n:00}_{(-Y,\uparrow)} \neq \mathbf{1} - \pi^{n:00}_{(-Y,\downarrow)}$	$1/6$	
$\mathbb{Q}_2$	$P(\uparrow,t)=1$	Contradicts $\mathbb{Q}_3$	1	$P(-Y,-W)=\frac{1}{12}$
$\mathbb{Q}_3$	$P(t,-W)=0$	Contradicts $\mathbb{Q}_2$	$1/2$	

Table 7: Table depicting the mathematical resolution of the AR after proper amendments.

4.5 Conclusion

The conclusion that the FRTE forces us to reject (Q), (C) or (S) is hence rendered moot. This is because the AR has other problems that can be the source of its erroneous zero probability result. Namely, it could be a (**1**) logical fallacy of equivocation or (**2**) a contradiction between two premises of its reasoning (with respect to whether some qubit has been measured or not) or (**3**) the failure to take into account the encoding of quantum information. All of which have been demonstrated. Ultimately, I think the main problem is really (**2**) as (**1**) and (**3**) can be seen as emergent errors from the fact that (**2**) exists. Thus, when I refer to the "fallacy in FR's AR" I will usually be referring to towards (**2**) to a greater extent than (**1**) and (**3**).

With this, we see that the FR argument is without a doubt in error. (Q), (C) and (S) do not need to be rejected by virtue of their thought experiment.

Now, I will expand on these findings.

5 Exploration & Expansion

In this section, I will show two avenues of expansion on the FRTE that can be seen as either critiques of the argument or simply as means to highlight or surface the fallacies employed in the AR.

5.1 FRTE-GHZ Thought Experiment

One of these avenues of expansion would be to make a FRTE version of the famous Greenberger-Horne-Zeilinger (GHZ) argument for nonlocality [23] [24] [25]. By doing this, we can see how the AR's fallacy (as surfaced in previous sections), can be used to also undermine the GHZ's conclusion for nonlocality. But first off, before trying to augment it, let us first do a brief review of the original argument.

5.1.1 The Original GHZ Argument: Set-Up

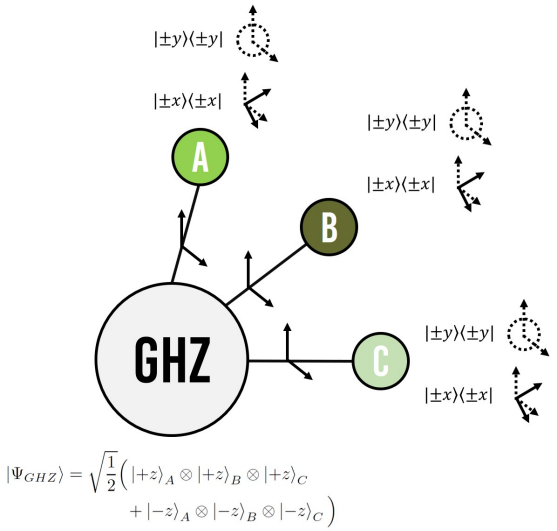


Figure 6: Pictorial of the Set-up for the GHZ experiment

While most experiments and arguments for nonlocality will conclude by some violation of a inequality, the GHZ argument is remarkable in the sense that it does not require any inequality. As depicted in the figure 6, this interesting feature arises primarily because of its use of three entangled systems rather than two. The overall wavefunction of these three entangled qubits are expressed as such:

$$|\Psi_{GHZ}\rangle = \sqrt{\frac{1}{2}}\Big(|+z\rangle_A \otimes |+z\rangle_B \otimes |+z\rangle_C + |-z\rangle_A \otimes |-z\rangle_B \otimes |-z\rangle_C\Big) \tag{25}$$

These qubits are then sent to three detectors, which can measure either in x-basis (that is, using the operator σ_x)

or y-basis (that is, σ_y). These just correspond respectively to the diagonal and circular bases of the qubit's original z-orientation.

$$|\pm x\rangle = \frac{1}{\sqrt{2}}\Big(|+z\rangle \pm |-z\rangle \Big) \tag{26}$$

$$|\pm y\rangle = \frac{1}{\sqrt{2}}\Big(|+z\rangle \pm i\,|-z\rangle \Big) \tag{27}$$

Consequently (and for the sake of reference):

$$\langle +z|\pm x\rangle = \frac{1}{\sqrt{2}} \quad , \qquad \langle -z|\pm x\rangle = \pm\frac{1}{\sqrt{2}} \quad , \qquad \langle +z|\pm y\rangle = \frac{1}{\sqrt{2}} \quad , \qquad \langle -z|\pm y\rangle = \pm\frac{i}{\sqrt{2}} \tag{28}$$

$$\langle \pm x|+z\rangle = \frac{1}{\sqrt{2}} \quad , \qquad \langle \pm x|-z\rangle = \pm\frac{1}{\sqrt{2}} \quad , \qquad \langle \pm y|+z\rangle = \frac{1}{\sqrt{2}} \quad , \qquad \langle \pm y|-z\rangle = \mp\frac{i}{\sqrt{2}} \tag{29}$$

Now with these standard preliminaries out of the way, let's say that each of these detectors correspond to an experimenter. Call them **Alice**, **Bob** and **Clare**. Now there's a total of 2^3 ways the three experimenters can choose their measurement devices, but the focus of the GHZ argument will be four of such combinations: xxx, xyy, yxy and yyx (where the position of each letter correspond to Alice, Bob and Clare respectively).

5.1.2 The Original GHZ Argument: Emergent Probabilities

Now, if we employ the same sort of notation discussed in subsection 4.2.1, we are able to obtain all 32 values of $P(\pm,\pm,\pm|^x/_y{}^x/_y{}^x/_y)$ for the aforementioned combinations. Consider three examples of this:

$$\texttt{Example 1}: P(+\,+\,+|xxx)$$

$$\begin{aligned}
P(+\,+\,+|xxx) &= \Big|\, \langle +x|_A \otimes \langle +x|_B \otimes \langle +x|_C \,|\Psi_{GHZ}\rangle \,\Big|^2 \\
&= \Big|\, \langle +x|_A \otimes \langle +x|_B \otimes \langle +x|_C \,\frac{1}{\sqrt{2}}\Big(|+z\rangle_A \otimes |+z\rangle_B \otimes |+z\rangle_C + |-z\rangle_A \otimes |-z\rangle_B \otimes |-z\rangle_C \Big) \Big|^2 \\
&= \frac{1}{2}\Big|\, \langle +x|_A \otimes \langle +x|_B \otimes \langle +x|_C \left(|+z\rangle_A \otimes |+z\rangle_B \otimes |+z\rangle_C \right) \\
&\quad + \langle +x|_A \otimes \langle +x|_B \otimes \langle +x|_C \left(|-z\rangle_A \otimes |-z\rangle_B \otimes |-z\rangle_C \right) \Big|^2 \\
\textbf{Noting } (29), \quad P(+\,+\,+|xxx) &= \frac{1}{2}\Big| \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \Big|^2 \\
&= \frac{1}{2}\Big| 2\cdot\frac{1}{2\sqrt{2}} \Big|^2 = \frac{1}{4}
\end{aligned} \tag{30}$$

Example 2 : $P(---|xxx)$

$$
\begin{aligned}
P(---|xxx) &= \Big|\langle -x|_A \otimes \langle -x|_B \otimes \langle -x|_C |\Psi_{GHZ}\rangle\Big|^2 \\
&= \Big|\langle -x|_A \otimes \langle -x|_B \otimes \langle -x|_C \frac{1}{\sqrt{2}}\Big(|+z\rangle_A \otimes |+z\rangle_B \otimes |+z\rangle_C + |-z\rangle_A \otimes |-z\rangle_B \otimes |-z\rangle_C\Big)\Big|^2 \\
&= \frac{1}{2}\Big|\langle -x|_A \otimes \langle -x|_B \otimes \langle -x|_C \Big(|+z\rangle_A \otimes |+z\rangle_B \otimes |+z\rangle_C\Big) \\
&+ \langle -x|_A \otimes \langle -x|_B \otimes \langle -x|_C \Big(|-z\rangle_A \otimes |-z\rangle_B \otimes |-z\rangle_C\Big)\Big|^2 \\
\textbf{Noting (29),}\quad P(---|xxx) &= \frac{1}{2}\Big|\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\Big|^2 \\
&= \frac{1}{2}\Big|0\Big|^2 \\
&= 0
\end{aligned}
\tag{31}
$$

Example 3 : $P(-+-|xyy)$

$$
\begin{aligned}
P(-+-|xyy) &= \Big|\langle -x|_A \otimes \langle +y|_B \otimes \langle -y|_C |\Psi_{GHZ}\rangle\Big|^2 \\
&= \Big|\langle -x|_A \otimes \langle +y|_B \otimes \langle -y|_C \frac{1}{\sqrt{2}}\Big(|+z\rangle_A \otimes |+z\rangle_B \otimes |+z\rangle_C + |-z\rangle_A \otimes |-z\rangle_B \otimes |-z\rangle_C\Big)\Big|^2 \\
&= \frac{1}{2}\Big|\langle -x|_A \otimes \langle +y|_B \otimes \langle -y|_C \Big(|+z\rangle_A \otimes |+z\rangle_B \otimes |+z\rangle_C\Big) \\
&+ \langle -x|_A \otimes \langle +y|_B \otimes \langle -y|_C \Big(|-z\rangle_A \otimes |-z\rangle_B \otimes |-z\rangle_C\Big)\Big|^2 \\
\textbf{Noting (29),}\quad P(-+-|xyy) &= \frac{1}{2}\Big|\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + (-\frac{1}{\sqrt{2}})(-\frac{i}{\sqrt{2}})\frac{i}{\sqrt{2}}\Big|^2 \\
&= \frac{1}{2}\Big|\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\Big|^2 \\
&= 0
\end{aligned}
$$

This can be replicated for all $P(\pm,\pm,\pm|^{x}/_{y}{}^{x}/_{y}{}^{x}/_{y})$ (this is not very tedious, by virtue of the symmetry of $|\Psi_{GHZ}\rangle$) and it will be found that they take the values of either $1/4$ or 0. I list them as follows:

Detector Combination	Probabilities of $1/4$			
xxx	$P(+++\|xxx)$	$P(+--\|xxx)$	$P(-+-\|xxx)$	$P(--+\|xxx)$
xyy	$P(++-\|xyy)$	$P(+-+\|xyy)$	$P(-++\|xyy)$	$P(---\|xyy)$
yxy	$P(++-\|yxy)$	$P(+-+\|yxy)$	$P(-++\|yxy)$	$P(---\|yxy)$
yyx	$P(++-\|yyx)$	$P(+-+\|yyx)$	$P(-++\|yyx)$	$P(---\|yyx)$

Detector Combination	All Zero Probabilities			
xxx	$P(++-\|xxx)$	$P(+-+\|xxx)$	$P(-++\|xxx)$	$P(---\|xxx)$
xyy	$P(+++\|xyy)$	$P(+--\|xyy)$	$P(-+-\|xyy)$	$P(--+\|xyy)$
yxy	$P(+++\|yxy)$	$P(+--\|yxy)$	$P(-+-\|yxy)$	$P(--+\|yxy)$
yyx	$P(+++\|yyx)$	$P(+--\|yyx)$	$P(-+-\|yyx)$	$P(--+\|yyx)$

These values emerge from the GHZ set-up and its wavefunction. The symmetries of these probabilities are unsurprising, as they are simply resultant from the greater symmetry found in $|\Psi_{GHZ}\rangle$.

Box 8: *Emergent Probabilities from the GHZ state and set-up.*

5.1.3 The Original GHZ Argument: Correlations

These emergent probabilities then result in one "correlation" per combination of detectors. If we let $r_i(\alpha)$ refer to the measurement outcome of some experimenter i measuring in α-basis (and α is either x or y), then we can see how certain correlations can be determined. Take for instance the detector combination xxx. Referring to Box 8, note how all the *non-zero* probabilities and, in turn, how all correlations $\Leftrightarrow r_A(x)r_B(x)r_C(x) = 1$ for any actualizable measurment outcome:

Probability	Correlation - $r_A(x)r_B(x)r_C(x)$
$P(+++\|xxx)$	$(+1)(+1)(+1) = 1$
$P(+--\|xxx)$	$(+1)(-1)(-1) = 1$
$P(-+-\|xxx)$	$(-1)(+1)(-1) = 1$
$P(--+\|xxx)$	$(-1)(-1)(+1) = 1$

Thus,

$$r_A(x)r_B(x)r_C(x) = 1 \tag{32}$$

always holds for all actualizable cases of the xxx detector configuration. Similarly, consider xxy,

Probability	Correlation - $r_A(x)r_B(y)r_C(y)$
$P(++-\|xyy)$	$(+1)(+1)(-1) = -1$
$P(+-+\|xyy)$	$(+1)(-1)(+1) = -1$
$P(-++\|xyy)$	$(-1)(+1)(+1) = -1$
$P(---\|xyy)$	$(-1)(-1)(-1) = -1$

In this way,

$$r_A(x)r_B(y)r_C(y) = -1 \tag{33}$$

holds for all actualizable measurements possibilities for the xyy detector configuration. By symmetry of the emergent probabilities, the following correlations can also be established:

$$r_A(x)r_B(x)r_C(x) = 1$$

$$r_A(x)r_B(y)r_C(y) = -1$$

$$r_A(y)r_B(x)r_C(y) = -1 \tag{34}$$

$$r_A(y)r_B(y)r_C(x) = -1 \tag{35}$$

Box 9: *Consequent Correlations from Emergent Probabilities*

5.1.4 The Original GHZ Nonlocality Argument

This is where the argument for nonlocality comes in. If the statistics of the $|\Psi_{GHZ}\rangle$ is determined by some local hidden variables, then there has to be some underlying predetermined strategy of sorts that govern the state of the particle where they are measured. This strategy will, naturally, need to encapsulate all the correlations, (32) to (35), in order to fulfill the statistics of the state at hand.

But does such a predetermined strategy exist? The answer is no. Consider how (32) and (33) give:

$$\begin{aligned} r_A(x)r_B(y)r_C(y) &= -r_A(x)r_B(x)r_C(x) \\ r_B(y)r_C(y) &= -r_B(x)r_C(x) \end{aligned} \tag{36}$$

In a similar vein, (34) and (35) give:

$$\begin{aligned} r_A(y)r_B(x)r_C(y) &= r_A(y)r_B(y)r_C(x) \\ r_B(x)r_C(y) &= r_B(y)r_C(x) \end{aligned} \tag{37}$$

Now if we divide (37) by (36):

$$\frac{r_B(x)r_C(y)}{r_B(y)r_C(y)} = \frac{r_B(y)r_C(x)}{-r_B(x)r_C(x)}$$
$$-r_B^2(x) = r_B^2(y)$$
$$\therefore r_B(x) = r_B(y) = 0 \tag{38}$$

But this result in (38) contradicts the set-up's nature as all $r_i(\alpha)$ must be ± 1. Thus, this proof by contradiction shows that our original assumption (that there exists some predetermined strategy or local hidden variables governing the behaviour of the particles) is a false one. Thus, a violation of locality is found.

In this way, one can summarize the argument as follows:

–	Propositional Calculus	Relevant Section / Equation
p_1	$\|\Psi_{GHZ}\rangle$	5.1.1 - See (25).
p_2	GHZ Set-Up	5.1.1 - Figure 7.
p_3	GHZ Set-Up $\rightarrow (r_i(\alpha) = \pm 1)$	5.1.1, 5.1.3.
c_1	$(r_i(\alpha) = \pm 1)$	From p_2, p_3 **MP**
p_4	(GHZ Set-Up $\wedge$ $\|\Psi_{GHZ}\rangle$) $\rightarrow$ Emergent Probabilities	5.1.2 - See Box 8.
p_5	Emergent Probabilities $\rightarrow$ Correlations	5.1.3 - See Box 9.
c_2	Correlations	From p_1, p_2, p_4, p_5 **MP,HS**
p_6	(Correlations $\wedge$ Locality) $\rightarrow (r_B(\alpha) = 0)$	5.1.4 - Proof: (36) to (38).
p_7	$(r_B(\alpha) = 0) \rightarrow \neg(r_i(\alpha) = \pm 1)$	Mathematically.
c_3	$\therefore \neg$(Locality)	From c_1, c_2, p_5, p_6 **HS,MT,CX**
c_f	$\therefore$ Nonlocality	From c_4

Deductive Argument 4: The Nonlocality Argument for GHZ Experiments

5.1.5 Approach & Set-Up of a GHZ version of FRTE

Now, how might the fallacy in FR's AR also be employed in a GHZ-like context? And in such a way that the above argument for nonlocality might be avoided? I suggest that this can be done by finding a way that the fallacy might help us reject premise p_4. One can do this by coming up with a new set of probabilities via an AR that mirrors the one in the original FRTE.

To do this, let's first combine the FRTE set-up with the GHZ experiment. Let there be 3 pairs of Wigner-Wigner's Friends: Alice, Bob and Clare (as "friends") in their labs and then **Pin**, **Qin** and **Rin** (as "Wigners") measuring those labs respectively.

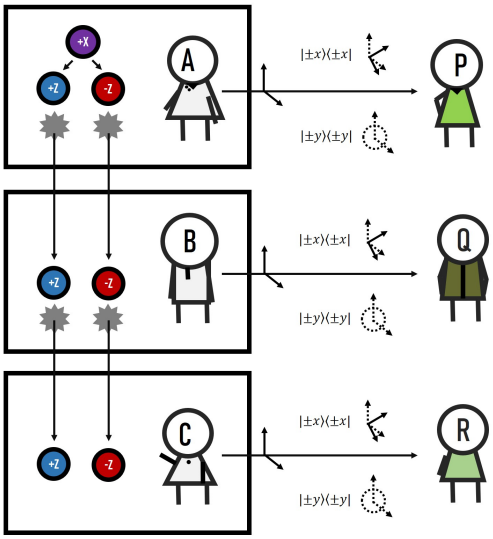


Figure 7: Pictorial of the Set-up for an FRTE-like GHZ experiment

Alice begins first with a $|+x\rangle_A$ qubit and measures it in $|\pm z\rangle$. Whatever state she measures she sends a corresponding particle to both Pin and Bob. Bob also measures in $|\pm z\rangle$ for his particle and then sends a corresponding particle of whatever he measures to Qin and Clare. Finally Clare also measures in $|\pm z\rangle$ and sends a corresponding particle to Rin. Notice then we also get a total wavefunction that corresponds to $|\Psi_{GHZ}\rangle$.

Now, Pin, Qin and Rin now take the roles of the experimenters at the detectors, measuring their respective particles either in x-basis or y-basis.

5.1.6 New Emergent Probabilities & the Alternative Reasoning

Now, with reference to (25), (28) and (29), and some calculations, one can see how:

$$\mathsf{Tr}\left[\rho_{GHZ,jk}\,|\pm^{x}/_{y}\rangle\langle\pm^{x}/_{y}|_{j}\,|\pm^{x}/_{y}\rangle\langle\pm^{x}/_{y}|_{k}\right]=\frac{1}{4} \tag{39}$$

Or, equivalently,

$$\mathsf{Tr}\left[\mathsf{Tr}_{i}\left[\,|\Psi_{GHZ}\rangle\,\langle\Psi_{GHZ}|\,\right]|\pm^{x}/_{y}\rangle\langle\pm^{x}/_{y}|_{j}\,|\pm^{x}/_{y}\rangle\langle\pm^{x}/_{y}|_{k}\right]=\frac{1}{4} \tag{40}$$

Where Tr_i refers to taking a partial trace over i, while i,j,k are any permutation of labs A,B,C. In other words, $P(\pm_j,\pm_k|^{x}/_{y}{}^{x}/_{y})=\frac{1}{4}$. Consider two examples of this:

Example 1 : $P(+_A -_B |xx)$

$$\begin{aligned}
P(+_A +_B |xx) &= \mathsf{Tr}\Big[\mathsf{Tr}_C\Big[\,|\Psi_{GHZ}\rangle\langle\Psi_{GHZ}|\,\Big]\,|+x\rangle\langle+x|_A\,|+x\rangle\langle+x|_B\Big] \\
&= \frac{1}{2}\cdot\mathsf{Tr}\Big[\big(\,|+z\rangle_A\,|+z\rangle_B + |-z\rangle_A\,|-z\rangle_B\,\big) \\
&\qquad\cdot\big(\,\langle+z|_A\,\langle+z|_B + \langle-z|_A\,\langle-z|_B\,\big)\big(\,|+x\rangle\langle+x|_A\,|+x\rangle\langle+x|_B\,\big)\Big] \\
\textbf{Consider: } |+x\rangle\langle+x|_A\,|+x\rangle\langle+x|_B &= \frac{1}{2}\frac{1}{2}\cdot\big(\,|+z\rangle_A + |-z\rangle_A\,\big)\big(\,\langle+z|_A + \langle-z|_A\,\big)\big(\,|+z\rangle_B + |-z\rangle_B\,\big)\big(\,\langle+z|_B + \langle-z|_B\,\big) \\
&= \frac{1}{4}\cdot\Big(\,|+z\rangle_A\,|+z\rangle_B\,\langle+z|_A\,\langle+z|_B + |+z\rangle_A\,|-z\rangle_B\,\langle+z|_A\,\langle-z|_B \\
&\qquad + |-z\rangle_A\,|+z\rangle_B\,\langle-z|_A\,\langle+z|_B\,|-z\rangle_A\,|-z\rangle_B\,\langle-z|_A\,\langle-z|_B\,\Big)
\end{aligned}$$

Hence, ignoring cross-terms:

$$\begin{aligned}
P(+_A +_B |xx) &= \frac{1}{8}\cdot\mathsf{Tr}\Big[\,|+z\rangle_A\,|+z\rangle_B\,\langle+z|_A\,\langle+z|_B + |-z\rangle_A\,|-z\rangle_B\,\langle-z|_A\,\langle-z|_B\,\Big] \\
&= \frac{1}{8}\cdot 2 = \frac{1}{4}
\end{aligned}$$

Example 2 : $P(-_A +_C |yx)$

$$\begin{aligned}
P(-_A +_C |yx) &= \mathsf{Tr}\Big[\mathsf{Tr}_B\Big[\,|\Psi_{GHZ}\rangle\langle\Psi_{GHZ}|\,\Big]\,|-y\rangle\langle-y|_A\,|+x\rangle\langle+x|_C\Big] \\
&= \frac{1}{2}\cdot\mathsf{Tr}\Big[\big(\,|+z\rangle_A\,|+z\rangle_C + |-z\rangle_A\,|-z\rangle_C\,\big) \\
&\qquad\cdot\big(\,\langle+z|_A\,\langle+z|_C + \langle-z|_A\,\langle-z|_C\,\big)\big(\,|-y\rangle\langle-y|_A\,|+x\rangle\langle+x|_C\,\big)\Big] \\
&= \frac{1}{2}\cdot\mathsf{Tr}\Big[\big(\,|+z\rangle_A\,|+z\rangle_C + |-z\rangle_A\,|-z\rangle_C\,\big) \\
&\qquad\cdot\big(\,\langle+z|_A\,|-y\rangle_A\,\langle+z|_C\,|+x\rangle_C + \langle-z|_A\,|-y\rangle_A + \langle-z|_C\,|+x\rangle_C\,\big)\cdot\langle-y|_A\,\langle+x|_C\,\Big] \\
&= \frac{1}{2}\cdot\mathsf{Tr}\Big[\big(\,|+z\rangle_A\,|+z\rangle_C + |-z\rangle_A\,|-z\rangle_C\,\big)\cdot\Big(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\Big)\cdot\langle-y|_A\,\langle+x|_C\,\Big] \\
&= \frac{1}{4}\cdot\mathsf{Tr}\Big[\big(\,|+z\rangle_A\,|+z\rangle_C + |-z\rangle_A\,|-z\rangle_C\,\big)\cdot\big(1+i\big)\cdot\langle-y|_A\,\langle+x|_C\,\Big] \\
&= \frac{1}{4}\cdot\mathsf{Tr}\Big[\big(\,|+z\rangle_A\,|+z\rangle_C + |-z\rangle_A\,|-z\rangle_C\,\big)\cdot\big(1+i\big)\cdot\frac{1}{2}\big(\,\langle+z|_A - i\,\langle-z|_A\,\big)\big(\,\langle+z|_C + \langle-z|_C\,\big)\Big]
\end{aligned}$$

Ignoring cross-terms:

$$\begin{aligned}
P(-_A +_C |yx) &= \frac{1}{8}\cdot\mathsf{Tr}\Big[\big(\,|+z\rangle_A\,|+z\rangle_C + |-z\rangle_A\,|-z\rangle_C\,\big)\cdot\big(1+i\big)\cdot\big(\,\langle+z|_A\,\langle+z|_C - i\,\langle-z|_A\,\langle-z|_C\,\big)\Big] \\
&= \frac{1}{8}\cdot\mathsf{Tr}\Big[\,|+z\rangle_A\,|+z\rangle_C\,\langle+z|_A\,\langle+z|_C\,\big(1+i\big) + |-z\rangle_A\,|-z\rangle_C\,\langle-z|_A\,\langle-z|_C\,\big(1+i\big)\big(-i\big)\Big] \\
&= \frac{1}{8}\cdot\big(1+i-i+1\big) = \frac{1}{4}
\end{aligned}$$

Such calculations can be replicated for all relevant probabilities for any pair of labs, regardless of what basis Pin, Qin and Rin decide to measure in and regardless of what measurement outcome they are looking for. And the result is always $\frac{1}{4}$. Thus:

$$P(\pm_j, \pm_k|^x/_y{}^x/_y) = \frac{1}{4} \tag{41}$$

Now, the agents Pin, Qin and Rin can then use these facts, combining it with the fallacy in FR's AR to undercut p_4 from the GHZ argument. How so? Consider $P(---|xxx)$. We have seen how the proper calculative reasoning shows us its value is 0. But we can dubiously argue that it is not. Let's say that Rin before measuring anything, considers to herself that given (41),

$$P(-_B, -_C|xx) = \frac{1}{4} \tag{42}$$

She then concludes that, since she has not measured yet, there is a $\frac{1}{4}$ chance that she will measure $|-x\rangle_C$ and Qin also measure $|-x\rangle_B$. She then steps into Qin's shoes (which corresponds to the $\mathbb{C}_i$ and $\mathbb{X}_i$ premises in FR's AR). Rin considers how Qin could also reason via (41) that:

$$P(-_A, -_B|xx) = \frac{1}{4} \tag{43}$$

Hence, Qin could conclude that she there is a $\frac{1}{4}$ chance that she measures $|-x\rangle_B$ and Pin also measures $|-x\rangle_A$. And this way Rin can, in the spirit of FR's AR, argue that since there is $\frac{1}{4}$ chance that Qin and herself both measure $|-x\rangle$ and there is also a $\frac{1}{4}$ chance that Qin and Pin measure $|-x\rangle$, it "only makes sense" that there is a total of $\frac{1}{16}$ of all of them measuring $|-x\rangle$ with respect to their own particles in the $|\Psi_{GHZ}\rangle$ state.

Thus, through this sort of reasoning:

$$\boxed{\text{By AR}}: \quad P(---|xxx) = \frac{1}{16}$$

5.1.7 Conclusions

One can see how a similar argument can be made by any of the agents and for any combination of $P(\pm, \pm, \pm|^x/_y{}^x/_y{}^x/_y)$. Such that,

$$\boxed{\text{By AR}}: \quad P(\pm, \pm, \pm|^x/_y{}^x/_y{}^x/_y) = \frac{1}{16} \tag{44}$$

This obviously stands in contradiction to the emergent probabilities we calculated in section 5.1.2. Thus, we have successfully undermined p_4 and escaped the nonlocality conclusion of delineated GHZ argument.

–	Propositional Calculus	Relevant Section / Equation
p_1	$\|\Psi_{GHZ}\rangle$	5.1.1 - See (25).
p_2	GHZ Set-Up	5.1.1 - Figure 7.
p_3	GHZ Set-Up $\rightarrow (r_i(\alpha) = \pm 1)$	5.1.1, 5.1.3.
c_1	$(r_i(\alpha) = \pm 1)$	From p_2, p_3 MP
×	(GHZ Set-Up $\wedge$ $\|\Psi_{GHZ}\rangle$) $\rightarrow$ Emergent Probabilities ×	5.1.5. Dubiously argued away by the AR fallacy.
p_5	Emergent Probabilities $\rightarrow$ Correlations	5.1.3 - See Box 9.
×	Correlations ×	From p_1, p_2, p_4 ×, p_5
p_6	(Correlations $\wedge$ Locality) $\rightarrow (r_B(\alpha) = 0)$	5.1.4 - Proof: (36) to (38).
p_7	$(r_B(\alpha) = 0) \rightarrow \neg(r_i(\alpha) = \pm 1)$	Mathematically.
×	$\therefore \neg$(Locality) ×	From c_1, c_2 ×, p_5, p_6
×	$\therefore$ Nonlocality ×	From c_4 ×

Deductive Argument 5: Deductive Argument 4 Dubiously Argued Away via AR Fallacy

One can see how this logical sleight of hand in such reasoning is really just a replication of the fallacy in FR's alternative route. Returning to the case of $P(---|xxx)$, here Rin's combination of those probabilities was based on a contradiction. Rin's consideration of $P(-_B, -_C|xx)$ requires Alice's lab to be *unmeasured* and thus at $|+z\rangle_A$. This fact is mathematically encapsulated in the partial trace over A during our calculations. Later, Rin imagines herself as Qin reasoning using $P(-_A, -_B|xx)$, she then adopts a conditional clause of Alice's lab being *measured* and at $|-x\rangle_A$. This is because $P(-_A, -_B|xx)$ is a *joint* probability of both Alice's and Bob's labs.

Hence, it is only by tying together these two contradictory conditonal clauses, Rin's AR gives rise to an argument against non-locality. The novel conclusion that we might make from combining GHZ and FRTE is that this same fallacy can be replicated in different contexts to cast doubt on otherwise sound arguments, highlighting the clearly mistaken nature of this little logical trick.

5.2 Reiterating the FR Alternative Reasoning Infinite Times

Recall that the no-go theorem suggests that the cause of the contradiction between the CR and the AR has to do with how (Q), (S) and (C) cannot be held together. I have argued that this is not the case. Rather, the contradiction emerges simply from a hidden contradiction (see section 4.3.4). These facts can be strengthened further if one can see how when this mistake is extended many iteration it leads to an absurdity. It seems that a viable way to do this is by considering a Schmidt decomposed expression of the FR state.

5.2.1 Schmidt Decomposition of the FR state

For any pure state of a bipartite system, $|\Phi\rangle$, there exists orthonormal states $|i\rangle_A$ for the former system A and orthonormal states $|i\rangle_B$ for the latter system B such that:[18]

$$\begin{aligned} |\Phi\rangle &= \lambda_0 |0\rangle_A \otimes |0\rangle_B + \lambda_1 |1\rangle_A \otimes |1\rangle_B + \cdots \lambda_\tau |\tau\rangle_A \otimes |\tau\rangle_B \\ &= \sum_{i=0}^{\tau} \lambda_i |i\rangle_A \otimes |i\rangle_B = \sum_{i=0}^{\tau} \lambda_i |ii\rangle \end{aligned} \tag{45}$$

Where λ_i are called "Schmidt" numbers or coeffecients and τ is number of degrees of freedom that the bases for the systems have. Naturally, by normalization $\sum_{i=0}^{\tau} \lambda_i^2 = 1$.

Since the FR state can also be construed as that of a bipartite system one can also find its Schmidt decomposition. This can be done explicitly but the point of the argument will not require the explicit values and vectors to be found. Hence, consider how the FR state can be expressed as a general Schmidt state:

$$\begin{aligned} |\Psi\rangle_{FR} &= \sqrt{\frac{1}{3}}\Big(|tT, \downarrow\Downarrow\rangle + |tT, \uparrow\Uparrow\rangle + |hH, \downarrow\Downarrow\rangle \Big) \\ &= \cos\theta \underbrace{\Big(\mathrm{a}\,|tT\rangle + \mathrm{b}\,|hH\rangle\Big)}_{|0\rangle_A} \otimes \underbrace{\Big(\mathrm{c}\,|\downarrow\Downarrow\rangle + \mathrm{d}\,|\uparrow\Uparrow\rangle\Big)}_{|0\rangle_B} \\ &+ \sin\theta \underbrace{\Big(\mathrm{b}^*\,|tT\rangle + \mathrm{a}^*\,|hH\rangle\Big)}_{|1\rangle_A} \otimes \underbrace{\Big(\mathrm{d}^*\,|\downarrow\Downarrow\rangle + \mathrm{c}^*\,|\uparrow\Uparrow\rangle\Big)}_{|1\rangle_B} \end{aligned} \tag{46}$$

And thus we can define the above expression according to a new basis and rewrite it as:

$$|\Psi\rangle_{FR} \quad \rightarrow \quad |\Psi\rangle_{\mathsf{SD}} = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle \tag{47}$$

Noting that due to the weights of $|\Psi\rangle_{FR}$, the Schmidt coeffecients $\cos\theta$ and $\sin\theta$ cannot be 0.

[18] A proof for this can be found in [27].

5.2.2 Reiterating the Alternate Reasoning Infinite Times

Now consider how FR'a AR can be employed with respect to $|\Psi\rangle_{\text{SD}}$. Let's say that Alice makes a measurement in some general basis, call it $|\texttt{A}\rangle_A$:

$$|\texttt{A}\rangle_A = \cos\alpha\,|0\rangle_A + \sin\alpha\,|1\rangle_A \tag{48}$$

It seems then we can once again do some roleplaying reasoning corresponding to $\mathbb{C}_i$ and $\mathbb{X}_i$. But rather than using the confusing language of Bob thinking as Alice thinking as Bob thinking as Alice and so on, let's just take it that Alice somehow "telephones" Bob (and vice versa) what they have measured and that they have an unwavering trust in each other's testimony and reasoning abilities. So, Alice phones Bob telling him that she received $|\texttt{A}\rangle_A$. Now Bob, trusting Alice, simply reasons through steering that this must mean that he is in $|\texttt{B-A}\rangle_B$ (obtained by finding $\langle\texttt{A}|_A\,|\Psi\rangle_{\text{SD}}$ and then normalizing):

$$|\texttt{B-A}\rangle_B = \frac{\cos\alpha\cdot\cos\theta\,|0\rangle_B + \sin\alpha\cdot\sin\theta\,|1\rangle_B}{\sqrt{\cos^2\alpha\cdot\cos^2\theta + \sin^2\alpha\cdot\sin^2\theta}} \tag{49}$$

This conclusion is made by virtue of the entanglement of the $|\Psi\rangle_{\text{SD}}$ state, Alice's claim to be in $|\texttt{A}\rangle_A$ *and* the AR's mistake. Hence, let n be the total number of times that this mistake is employed by both Alice and Bob.

Now, at this point, Bob phones Alice in return, telling her that he has inferred that he is at $|\texttt{B-A}\rangle_B$. Hearing this. Alice, via the AR fallacy, reasons that she must be mistaken about her earlier measurement. Therefore, she takes herself to be in $|\texttt{A-B-A}\rangle_A$ as expressed below, performing the exact same sort of AR fallacy from previously.

$$\boxed{n=1}: \quad |\texttt{A-B-A}\rangle_A = \frac{\cos\alpha\cdot\cos^2\theta\,|0\rangle_A + \sin\alpha\cdot\sin^2\theta\,|1\rangle_A}{\sqrt{\cos^2\alpha\cdot\cos^4\theta + \sin^2\alpha\cdot\sin^4\theta}} \tag{50}$$

Where n is the total number of times that this mistake is employed by both Alice and Bob. Notice, now one can repeat this indefinitely. Alice then updates Bob. Bob then updates Alice. Hence, each of them will get a general state of:

$$\boxed{n=N}: \quad |N\rangle_{A/B} = \frac{\cos\alpha\cdot\cos^{N+1}\theta\,|0\rangle_{A/B} + \sin\alpha\cdot\sin^{N+1}\theta\,|1\rangle_{A/B}}{\sqrt{\cos^2\alpha\cdot\cos^{2N+2}(\theta) + \sin^2\alpha\cdot\sin^{2N+2}(\theta)}} \tag{51}$$

Where all odd n will register for Bob and all even n for Alice. One can see how then such a state *converges*. As Alice and Bob phone each other back and forth onward into eternity, the state that each of them find themselves in will converge depending on θ, such that:

$$\begin{aligned}\cos\theta > \sin\theta &\Rightarrow |N=\infty\rangle_{A/B} = |0\rangle_{A/B}\\ \cos\theta < \sin\theta &\Rightarrow |N=\infty\rangle_{A/B} = |1\rangle_{A/B}\end{aligned} \tag{52}$$

Noting that $\sin\theta \neq \cos\theta$, since $|\Psi\rangle_{\text{SD}}$ is not maximally entangled. Consequently, this means the *joint* wavefunction $|\Psi\rangle_{\text{SD}}$ would also converge:

$$\begin{aligned}\cos\theta > \sin\theta &\Rightarrow |\Psi\rangle_{\text{SD}} = |00\rangle\\ \cos\theta < \sin\theta &\Rightarrow |\Psi\rangle_{\text{SD}} = |11\rangle\end{aligned} \tag{53}$$

Thus,

$$|\Psi\rangle_{\text{SD}} = |00\rangle \quad \oplus^{19} \quad |11\rangle \tag{54}$$

5.2.3 Conclusion

This then provides a kind of proof by (inspired in part by "Hardy's Ladder" [26]) contradiction against the fallacy of the AR, since it gives a conclusion mutually exclusive with the fact that $|\Psi\rangle_{\text{SD}}$ is a superposition of $|00\rangle$ *and* $|11\rangle$ as in (47).

I provide here a syllogism to illustrate the flow of my argument:

–	Propositional Calculus	Relevant Section / Equation
p_1	$\|\Psi\rangle_{\text{SD}}$	(45) to (47).
p_2	$\|\Psi\rangle_{\text{SD}} = \cos\theta\,\|00\rangle + \sin\theta\,\|11\rangle$	(47).
p_3	$(\|\Psi\rangle_{\text{SD}} \wedge (\text{F})^{\infty}) \;\rightarrow\; (\|\Psi\rangle_{\text{SD}} = \|00\rangle \oplus \|11\rangle)$	5.2.2, (F) denotes the AR fallacy.
p_4	$(\|\Psi\rangle_{\text{SD}} = \|00\rangle \oplus \|11\rangle)$ $\rightarrow\; \neg(\|\Psi\rangle_{\text{SD}} = \cos\theta\,\|00\rangle + \sin\theta\,\|11\rangle)$	By inspection, noting that Schmidt numbers of $\|\Psi\rangle_{FR}$ are not zero.
c_1	$\neg(\|\Psi\rangle_{\text{SD}} \wedge (\text{F})^{\infty})$	From p_2, p_3, p_4 **MT,MT**
c_2	$\neg((\text{F})^{\infty})$	From c_1, p_1 **CX**
c_3	$\neg(\text{F})$	From c_2
c_f	The AR fallacy is shown as mistaken.	From c_3

Deductive Argument 6: Outline of the Argument for Section 5.2

[19] Note that this $\oplus$ symbol here denotes an exclusive disjunction. Also, since there is only one unique Schmidt decomposition, one could in principle find which one of these concludes the infinite number of iterations. However, for the sake of this argument, finding this result is not necessary. This might be useful as this will generalize for *any pure state* for a bipartite system.

6 Review & Deductive Logic Analysis

As mentioned in subsection 4.1, this part of the paper is a kind of preliminary before the arguments made in section 4. Here I will more closely and tightly examine the deductive structure of the FR paper and show how I go about refuting their no-go theorem.

6.1 FR's Argument as Propositional Calculus

–	Propositional Calculus	Relevant Section / Equation
p_1	AR $\rightarrow (P(-Y,-W)=0)$	3.1.2, Box 1, (13)
c_1	$\neg(P(-Y,-W)=0) \rightarrow \neg$(AR)	From p_1 **CP**
p_2	CR $\rightarrow= \neg(P(-Y,-W)=0)$	3.1.1, (10), $(P(-Y,-W)=\frac{1}{12}) \rightarrow \neg(P(-Y,-W)=0)$.
p_3	CR	Calculative reasoning clearly correct.
c_2	$\neg(P(-Y,-W)=0)$	From p_2, p_3 **MP**
c_3	$\neg$(AR)	From c_1, c_2 **MP**
p_4	AR $\Leftrightarrow$ ((Q) $\wedge$ (C) $\wedge$ (S))	[1], Box 2.
c_4	$\neg$((Q) $\wedge$ (C) $\wedge$ (S))	From c_3, p_4 **MT**
c_5	$\neg$(Q) $\vee \neg$(C) $\vee \neg$(S)	From c_4, the FR no-go theorem
c_f	FR No-go Theorem	From c_5, $\neg$(Q) $\vee \neg$(C) $\vee \neg$(S) $\Leftrightarrow$ FR No-go Theorem

Deductive Argument 7: The FR Argument for the No-Go Theorem

As c_f states, the conclusion of c_5 "$\neg$(Q) $\vee \neg$(C) $\vee \neg$(S)", translates to the idea that we must at least give up (Q), (C) or (S). Which corresponds to the no-go theorem that FR postulates.

See how then that saying that the AR is wrong and that the CR is correct does nothing to refute the no-go theorem, since these affirm premises of FR's own argument (namely c_3 and p_3 respectively). I have suggested (as in Box 4) that the best way to avoid the final conclusion c_f is to look to denying p_4. The only way to do this is to find if all the steps of the AR map over nicely to these three major assumptions (Q), (C) and (S).

To do this, naturally, will require a good understanding of both the AR and the three assumptions. Thus, it is good to reduce these to their logical deductive raw, so to speak. I will do this for the three assumptions before the AR.

6.2 FR's (Q), (C) & (S) Analyzed

Here I do something more "exegetical" to get a good sensing of what FR means by (Q), (S) and (C) so that we can give a charitable and accurate analysis of the AR later and see if the aforemention premise p_4 holds true or not. I will do this in point form:

- (Q) $\Leftrightarrow$ "Quantum Born Rule"
  - In FR's words: "that an agent can be certain that a given proposition holds whenever the quantum-mechanical Born rule assigns probability 1 to it".
  - FR claims that this assumption encompasses the use of quantum born rule during the reasoning of each agent. In the paper, they describe the reasoning process of (Q) within each agent in terms of Heisenberg projectors which further break down to state projections and isometries. Consequently, a closer look to these projections and isometries will be a important avenue to evaluate FR's argumentation.
- (C) $\Leftrightarrow$ "Consistency"
  - It is particularly tough to understand what FR tries to capture in this particular aspect of the no-go theorem since their descriptions of them are (ironically) not very consistent. For instance:
    1. **Page 2** of their paper states that (C) demands that "predictions" of different agents under the same theoretical framework "are not contradictory". This should roughly translate to the idea that
       - If agent$_1$[20] is certain of some result $|\alpha\rangle$[21] then $\Rightarrow$ agent$_2$ is certain that $|\alpha\rangle$ was the result measured.
    2. Meanwhile, **page 5** states that (C) just means
       - **If agent$_2$ is certain that** agent$_1$ is certain of that result $|\alpha\rangle$ then $\Rightarrow$ agent$_2$ is certain that $|\alpha\rangle$ was the result measured.
  - With some inspection, we see how these are actually two different sorts of claims. Specifically the claim from **page 2** requires an additional clause[22] to promote itself to **page 5**'s claim.
  - For now, we will take (C) to be the more complete claim taken from **page 5**.
  - Do note that a negation of this is not to merely say that one agent has made an error in his reasoning process. Rather, it is to say that two agents following the exact same theoretical framework and every step can still arrive at mutually exclusive results purely because the theory being employed cannot extend across agents using it.
  - A negation of (C) is what FR takes as "quantum theory cannot describe the use of itself".
- (S) $\Leftrightarrow$ "Single-Outcome"
  - In FR's words: "from the viewpoint of an agent who carries out a particular measurement, this measurement has one single outcome."
  - Essentially: If I am certain that p[23] then $\Rightarrow$ I am certain of *the denial of* $\neg p$. (Yes, it is that trivial.)

[20] Who is using the same theoretical framework as agent$_2$.
[21] Specifically that some result $|\alpha\rangle$ was measured at time t.
[22] That is "If agent$_2$ is certain that".
[23] That is, some proposition p. For instance "$|\alpha\rangle$ was measured at time t".

6.3 The Alternative Reasoning as Propositional Calculus

Now we move on to the next part: breaking the AR into propositional calculus so as to do the comparison. From the provisional sketch provided in Box 1, we can see how the AR roughly runs. For FR, the AR is delineated by their paper's **Table 3**. Here, I produce a copy of the table, but with the notation I am employing:

Agents	Assumed Observation	"Inferred via (Q)"	"Further Implied"	"Inferred via (C)"
Alice	T4 Alice measures $\|t\rangle_A \otimes \|T\rangle_{P_A}$ at n:01	T5 Alice is certain that Wigner will observe $\|+W\rangle$ at n:31	-	-
Bob	T2 Bob measures $\|\uparrow\rangle_B \otimes \|\Uparrow\rangle_{P_B}$ at n:11	T3 Bob is certain that Alice knows she measured $\|t\rangle_A \otimes \|T\rangle_{P_A}$ at n:01	T6 Bob is certain that Alice is certain that Wigner will observe $\|+W\rangle$ at n:31	T7 Bob is certain that Wigner will observe $\|+W\rangle$ at n:31
Yvette	T0 Yvette measures $\|-Y\rangle$ at n:21	T1 Yvette is certain that Bob knows he measured $\|\uparrow\rangle_B \otimes \|\Uparrow\rangle_{P_B}$ at n:11	T8 Yvette is certain that Bob is certain that Wigner will observe $\|+W\rangle$ at n:31	T9 Yvette is certain that Wigner will observe $\|+W\rangle$ at n:31
-	T10 Hence $P(-Y,-W)=0$			

Table 8: A Copy of FR's **Table 3**, in my notation.

Notice that the various Ti statements included for each cell here. From pages 4-8 of their paper, FR run through all these statements in the table in a pretty convoluted way (at least for me). But with some analysis, we can see how the table's statements logically bridge together from i $= 0$ to i $= 10$ for each Ti, leading Yvette to conclude that "$P(-Y,-W)=0$". If this way of linking the statements really goes as I have suggested, then we can illustrate them as such in Figure 8 included below.

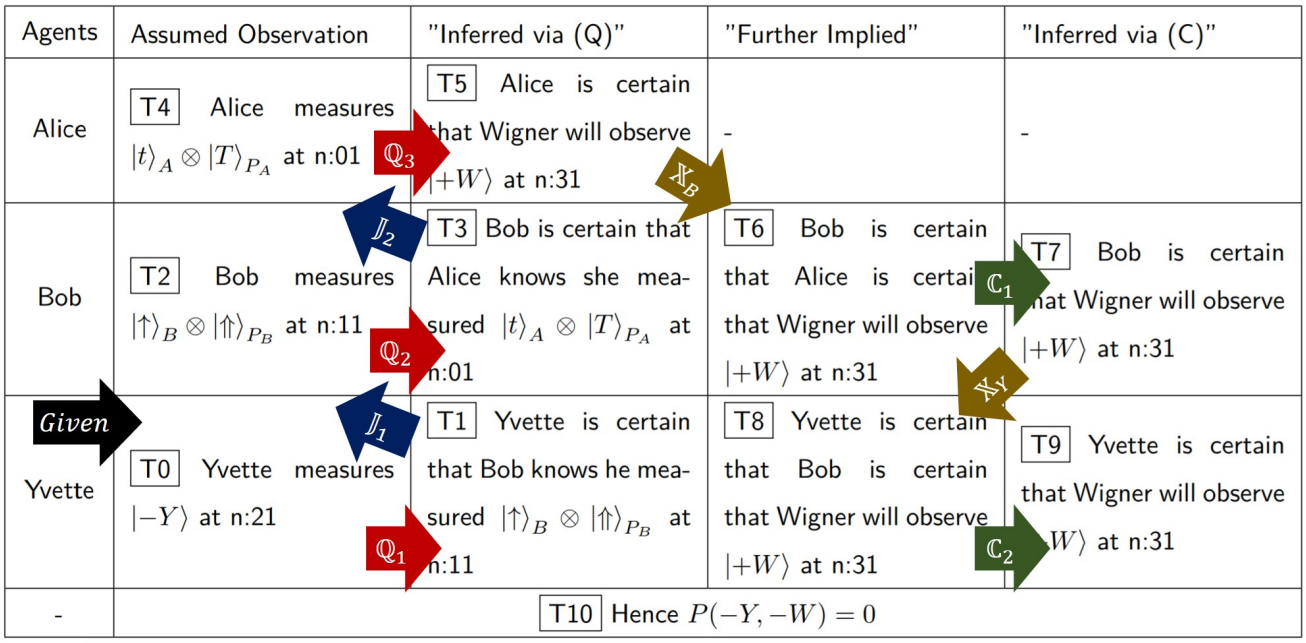


Figure 8: Diagram depicting the logical links between each cell in FR's **Table 3**

Hence, from this, we can decompose the table (and hence FR's AR) into propositional calculus, so as to see if all these premises really map over strictly to FR's (Q), (C) and (S) assumptions. My proposed decomposition[24] is as such:

[24]Note that the various $\mathbb{Q}_i$ and $\mathbb{C}_i$ do not necessarily map over to (Q) and (C) despite being the same alphabet. In fact, the purpose of this tedious conversion (of turning FR's **Table 3** into propositional calculus) is just so to check if (Q), (C) and (S) map over well to all the distinct premises required in their reasoning.

-	Class	Relevant Section / Table	Logical Calculus
p_1	-	Given	T0
p_2	$\mathbb{Q}_1$	Table 5, Section 6.4.3	T0 $\rightarrow$ T1
p_3	$\mathbb{J}_1$	Table 11, Section 6.4.2	T1 $\rightarrow$ T2
p_4	$\mathbb{Q}_2$	Table 5, Section 6.4.3	T2 $\rightarrow$ T3
p_5	$\mathbb{J}_2$	Table 11, Section 6.4.2	T3 $\rightarrow$ T4
p_6	$\mathbb{Q}_3$	Table 5, Section 6.4.3	T4 $\rightarrow$ T5
p_7	$\mathbb{X}_B$	Table 9, Section 6.4.1	T5 $\rightarrow$ T6
c_1		From p_5, p_6, p_7 **HS**$\times$**3**	$\therefore$ T3 $\rightarrow$ T6
p_8	$\mathbb{C}_1$	Table 10, Section 6.4.1	T6 $\rightarrow$ T7
p_9	$\mathbb{X}_Y$	Table 9, Section 6.4.1	T7 $\rightarrow$ T8
c_2		From p_3, p_4, c_1, p_8, p_9 **HS**$\times$**4**	$\therefore$ T1 $\rightarrow$ T8
p_{10}	$\mathbb{C}_2$	Table 10, Section 6.4.1	T8 $\rightarrow$ T9
p_{11}	-	Math	T9 $\wedge$ T0 $\rightarrow$ T10
c_f		From p_1, p_2, c_2, p_{10}, p_{11} **HS**,**MP**$\times$**3**	$\therefore$ T10

Deductive Argument 8: FR's Alternative Reasoning as a Propositional Calculus

Examining the above propositional calculus, we see that the argument is deductively airtight (the premises lead to the conclusion). One can see how the premises P1, P11 are not in dispute (the former is a given while the latter is just a mathematical step.) The only issues will be the various $\mathbb{Q}_i, \mathbb{J}_i, \mathbb{X}_i, \mathbb{C}_i$ premises. From the critical question (see Box 4), if these premises combined constitute anything beyond FR's (Q), (C) and (S), then the AR can be shown to be faulty without falling into no-go theorem's difficult trichotomy. It is to this we now turn to.

6.4 Does $\mathbb{Q}_i, \mathbb{J}_i, \mathbb{X}_i, \mathbb{C}_i$ consist of only (Q), (C) and (S)?

6.4.1 Analysis of $\mathbb{X}_i$ and $\mathbb{C}_i$

Let's first look at $\mathbb{X}_i$, which consists of the following two premises in the propositional calculus:

	Designation	In Propositional Calculus	In English
p_7	$\mathbb{X}_B$	T5 $\rightarrow$ T6	If Alice is certain that "*Wigner will observe* $\lvert -W\rangle$ *at n:31*" then Bob is certain Alice certain that "*Wigner will observe* $\lvert -W\rangle$ *at n:31*"
p_9	$\mathbb{X}_Y$	T7 $\rightarrow$ T8	If Bob is certain that "*Wigner will observe* $\lvert -W\rangle$ *at n:31*" then Yvette is certain Bob is that "*Wigner will observe* $\lvert -W\rangle$ *at n:31*"

Table 9: The two $\mathbb{X}_i$ premises and their place in the AR tabulated.

Note how both of the $\mathbb{X}_i$ premises consist of the following form: "If agent$_1$ is certain of proposition p then agent$_2$ is certain that agent$_1$ is certain of proposition p"

Moving on to $\mathbb{C}_i$, which consists of the following two premises in the propositional calculus:

	Designation	In Propositional Calculus	In English
p_8	$\mathbb{C}_1$	T6 $\rightarrow$ T7	If Bob is certain Alice is certain that *Wigner will observe* $\lvert -W\rangle$ *at n:31* then Bob is certain that *Wigner will observe* $\lvert -W\rangle$ *at n:31*
p_{10}	$\mathbb{C}_2$	T8 $\rightarrow$ T9	If Yvette is certain Bob is certain that *Wigner will observe* $\lvert -W\rangle$ *at n:31* then Yvette is certain that *Wigner will observe* $\lvert -W\rangle$ *at n:31*

Table 10: The two $\mathbb{C}_i$ premises and their place in the AR tabulated.

Note how both of the $\mathbb{C}_i$ premises consist of the following form: "If agent$_2$ is certain that agent$_1$ knows p then agent$_2$ is certain of proposition p"

Hence, when taken together, $\mathbb{X}_i$ and $\mathbb{C}_i$ corresponds quite well to FR's (C) criteria as discussed in section (6.2). Also, quite trivially, one can see how every single premise made above in the syllogism requires (S), as each premise assumes only one possible outcome and the law of non-contradiction.

6.4.2 Analysis of $\mathbb{J}_i$

We now turn our attention to $\mathbb{J}_i$, which consists of the following two premises in the propositional calculus:

	Designation	In Propositional Calculus	In English
p_3	$\mathbb{J}_1$	$\boxed{\text{T1}} \rightarrow \boxed{\text{T2}}$	If Yvette is certain Bob knows $\left\|\uparrow\right\rangle_B \otimes \left\|\Uparrow\right\rangle_{P_B}$ was measured at n:11 then $\left\|\uparrow\right\rangle_B \otimes \left\|\Uparrow\right\rangle_{P_B}$ was measured at n:11
p_5	$\mathbb{J}_2$	$\boxed{\text{T3}} \rightarrow \boxed{\text{T4}}$	If Bob is certain Alice knows $\left\|t\right\rangle_A \otimes \left\|T\right\rangle_{P_A}$ was measured at n:01 then $\left\|t\right\rangle_A \otimes \left\|T\right\rangle_{P_A}$ was measured at n:01

Table 11: The two $\mathbb{J}_i$ premises and their place in the AR tabulated.

Note how both of the $\mathbb{J}_i$ premises consist of the following form: "If agent$_1$ is certain that agent$_2$ is certain of proposition p then p". This can be further broken down to two components:

1. If agent$_1$ is certain that agent$_2$ knows p then agent$_2$ knows p
2. If agent$_2$ knows p then p

The second component is quite uncontroversial, since it is quite absurd to say I know that something but that something is not actually true. The first component on the other hand is something quite dependent on the *reasons* agent$_1$ has for being certain that agent$_2$ knows that proposition p. From its context in Deductive Argument 8, we see the reason that each of the agents (namely Bob and Yvette) is certain that their counterparts (Alice and Bob respectively) lies in the preceding premises $\mathbb{Q}_1$ and $\mathbb{Q}_2$ respectively. In other words, if $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are both true premises, then we have good grounds to also think $\mathbb{J}_1$ and $\mathbb{J}_2$.

Thus, it seems the truth value of $\mathbb{J}_i$ are dependent on their preceding $\mathbb{Q}_i$, making them of little interest. Combined with what we have seen in the previous section, we thus must pay special attention to the final set of premises $\mathbb{Q}_i$. This is because if all $\mathbb{Q}_i$ premises map over to (Q) then the no-go theorem is unavoidable. However, if there is something more to $\mathbb{Q}_i$ than the experimenters using the Born rule (as in (Q)), then the no-go theorem collapses.

6.4.3 Analysis of $\mathbb{Q}_i$

From the propositional calculus, we see three premises are categorized under $\mathbb{Q}_i$. These are described in detail in Table 5. To begin our analysis, consider again the wavefunctions that describe the relevant steps:

From n:10 to n:20, copying (5) here:

$$\sqrt{\frac{1}{3}}\left|t\right\rangle_A \otimes \left|T\right\rangle_{P_A} \otimes \left|\uparrow\right\rangle_B \otimes \left|\Uparrow\right\rangle_{P_B} + \sqrt{\frac{2}{3}}\left|+Y\right\rangle \otimes \left|\downarrow\right\rangle_B \otimes \left|\Downarrow\right\rangle_{P_B}$$

From n:00 to n:10, copying (4) here:

$$\left(\sqrt{\frac{2}{3}}\left|t\right\rangle_A \otimes \left|T\right\rangle_{P_A} \otimes \left|\rightarrow\right\rangle_B + \sqrt{\frac{1}{3}}\left|h\right\rangle_A \otimes \left|H\right\rangle_{P_A} \otimes \left|\downarrow\right\rangle_B\right) \otimes \left|\cdot\right\rangle_{P_B}$$

For $\mathbb{Q}_1$, because she has measured $|-Y\rangle$, Yvette simply infers that the second term cancels out, leaving behind $|\uparrow\rangle_B \otimes |\Uparrow\rangle_{P_B}$. Hence, naturally, she is "certain Bob knows he measured $|\uparrow\rangle_B \otimes |\Uparrow\rangle_{P_B}$ at n:11". Hence, FR correctly point out this is simply Yvette using the quantum born rule properly.

Likewise, for $\mathbb{Q}_2$, Bob rightly infers via quantum born rule that since he had measured $|\uparrow\rangle_B \otimes |\Uparrow\rangle_{P_B}$, Alice could only have measured $|t\rangle_A \otimes |T\rangle_{P_A}$, since it is only $|t\rangle_A \otimes |T\rangle_{P_A}$ that would have contributed a $|\uparrow\rangle_B \otimes |\Uparrow\rangle_{P_B}$ component via Alice sending a $|\rightarrow\rangle_B$ to Bob.[25]

So far, the no-go theorem still holds! We hence look to the final last aspect of $\mathbb{Q}_i$. Let's consider $\mathbb{Q}_3$:

> $\mathbb{Q}_3 \Leftrightarrow$ "**If** Alice measures $|t\rangle_A \otimes |T\rangle_{P_A}$ at n:01
> **then** she is certain that 'Wigner will observe $|+W\rangle$ at n:31'"

With some inspection of Table 5, we might notice something peculiar. While $\mathbb{Q}_1$ and $\mathbb{Q}_2$ concern agents reasoning backward in time by a step (specifically from n:21 to n:11 and from n:11 to n:01 respectively), $\mathbb{Q}_3$ reasons forward by by three steps (from n:01 to n:31). This in mind, we now ask: is $\mathbb{Q}_3$ simply Alice using the Born rule?

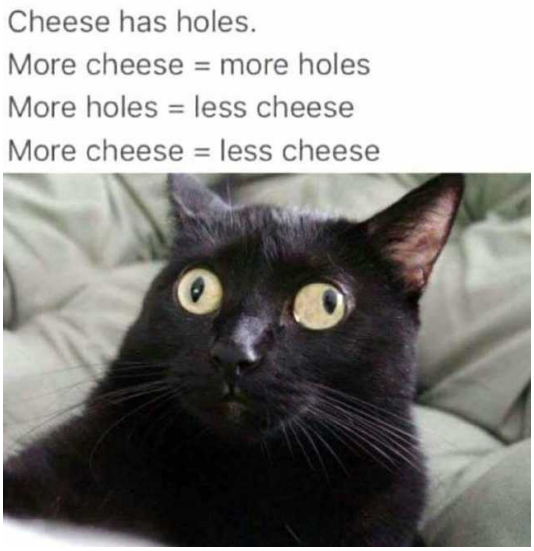
At first glance, the answer seems to be yes. But let us remember that Alice is using the Born rule forward in time to make predictions about the result Wigner will acquire at the end of the experiment. From the provisional sketch provided in Box 1, Alice's reasoning with Born rule is to note that since she is sending out a $|\rightarrow\rangle_B$ to Bob, it would correspond to $|+W\rangle$ hence rendering $|-W\rangle$ an impossible result. But are there any other hidden assumptions here?

[25] The exact mathematical steps are discussed in section 4.3

How is Alice sure that *no changes have been done to the set-up* between n:01 and n:31 (i.e her sending $|{\rightarrow}\rangle_B$ to Bob and Wigner's measurement of Bob's lab)?

Alright, now that we have done all the homework we have found a possible loophole to explore. Namely $\mathbb{Q}_3$ requires Alice to assume that the wavefunction has been the same from n:01 to n:31. And that is obviously a moot point. And thus, this is where the critiques from section 4 come in. And indeed we find that holding $\mathbb{Q}_3$ while holding to $\mathbb{Q}_2$ is indeed more than just mere Born rule calculations. It is to affirm to contradictory statements about whether Bob's received $|{\rightarrow}\rangle_B$ qubit has been measured or not.

7 Conclusions



Internet Meme 1: This is, hopefully, relevant.

In this paper I sought to show that all the buzz around the FRTE - be it that quantum mechanics is self-inconsistent or classical logic must be abandoned when studying quantum objects or relativism with regard to laboratory results - that all these are just scientific fake news. I showed this by analyzing closely the FR argument, their AR route and showed that it seems that the premise $\mathbb{Q}_3$ ("If Alice measures $|t\rangle_A \otimes |T\rangle_{P_A}$ at n:01 then she is certain that 'Wigner will observe $|{+}W\rangle$ at n:31'") goes beyond just the quantum Born rule assumption (Q), particularly when we also affirm premises like $\mathbb{Q}_1$ and $\mathbb{Q}_2$. This is because it will result in affirmation of a contradiction with respect to whether Bob's received qubit has been measured by Bob or not.

I have shown how this fallacy can be illustrated in three ways as discussed in section 4. I expanded on this by illustrating how this fallacy produces fallacious conclusions in GHZ context and also via Schmidt Decomposition, further highlighting the fact that this fallacy is what has gone wrong in the AR.

It should be clear then that I am not offering this fallacy as *just another* assumption to consider giving up *in addition to*

(Q) (S) and (C). Rather, I have sought to argue for a stronger conclusion: that the argument lends no weight against (Q), (S) and (C) *at all*, since it really is just due to a contradiction in its works.

To say that the FR argument still has some bite against the three assumptions would be as good as saying the famous internet "more cheese = less cheese" puzzle (see Internet Meme 1) lends evidence that against the existence of cheese. Sure, one can just deny the existence of cheese and be done with the puzzle, but that would be, dare I say it, bogus.[26] It is the same with the FR problem. The real issue is just bad logic and nothing else. Contradiction in, contradiction out - no surprises here! Hence, there is absolutely reason to give up any of the (Q), (S) or (C), nor punt to the existence of a multiverse or to resort to abandoning modal logic.

With this I add one last deductive argument to illustrate what I am saying, and hope that my treatment will be both seen as fair and not in vain. For the former, I have sought to do what I can. As for the latter, we can only wait and see.

–	Propositional Calculus	Relevant Section / Equation
p_1	AR $\rightarrow (P(-Y,-W)=0)$	3.1.2, Box 1, (13)
c_1	$\neg(P(-Y,-W)=0) \rightarrow \neg$(AR)	From p_1 **CP**
p_2	CR $\rightarrow= \neg(P(-Y,-W)=0)$	3.1.1, (10), $(P(-Y,-W)=\frac{1}{12})$ $\rightarrow \neg(P(-Y,-W)=0)$.
p_3	CR	Calculative reasoning clearly correct.
c_2	$\neg(P(-Y,-W)=0)$	From p_2, p_3 **MP**
c_3	$\neg$(AR)	From c_1, c_2 **MP**
×	AR $\Leftrightarrow$ ((Q) $\wedge$ (C) $\wedge$ (S)) ×	Refuted through this paper: AR contains a contradiction about Bob's qubit. See Deductive Argument 10.
×	$\neg$((Q) $\wedge$ (C) $\wedge$ (S)) ×	From c_3, p_4 ×
×	$\neg$(Q) $\vee \neg$(C) $\vee \neg$(S) ×	From c_4, ×
×	FR No-go Theorem ×	From c_5 ×, $\neg$(Q) $\vee \neg$(C) $\vee \neg$(S) $\Leftrightarrow$ FR No-go Theorem

Deductive Argument 9: The FR Argument for the No-Go Theorem Refuted

[26] There have been proposed solutions to this little internet meme by Reddit-dwelling philosophers of the internet. One of them would be that we have committed the fallacy of equivocation (much like in section 4.2). The cheese in "more cheese" is cheese as a whole object, which has holes. Meanwhile cheese in "less cheese" is cheese as a substance, which by definition has no holes.

-	Class	Relevant Section / Table	Logical Calculus	Remarks
p_1	-	Given	T0	
p_2	$\mathbb{Q}_1$	Table 5, Section 6.4.3	T0 $\rightarrow$ T1	Equivocation fallacy with other $\mathbb{Q}_i$.
p_3	$\mathbb{J}_1$	Table 11, Section 6.4.2	T1 $\rightarrow$ T2	
p_4	$\mathbb{Q}_2$	Table 5, Section 6.4.3	T2 $\rightarrow$ T3	Requires Bob's qubit to be measured at $\|\uparrow\rangle_B$. See (21).
p_5	$\mathbb{J}_2$	Table 11, Section 6.4.2	T3 $\rightarrow$ T4	
p_6	$\mathbb{Q}_3$	Table 5, Section 6.4.3	T4 $\rightarrow$ T5	Requires Bob's qubit to be not measured at $\|\rightarrow\rangle_B$. See (22).
p_7	$\mathbb{X}_B$	Table 9, Section 6.4.1	T5 $\rightarrow$ T6	
c_1		From p_5, p_6, p_7 HS$\times$3	$\therefore$ T3 $\rightarrow$ T6	
p_8	$\mathbb{C}_1$	Table 10, Section 6.4.1	T6 $\rightarrow$ T7	
p_9	$\mathbb{X}_Y$	Table 9, Section 6.4.1	T7 $\rightarrow$ T8	
c_2		From p_3, p_4, c_1, p_8, p_9 HS$\times$4	$\therefore$ T1 $\rightarrow$ T8	
p_{10}	$\mathbb{C}_2$	Table 10, Section 6.4.1	T8 $\rightarrow$ T9	
p_{11}	-	Math	T9 $\wedge$ T0 $\rightarrow$ T10	
c_f		From p_1, p_2, c_2, p_{10}, p_{11} HS,MP$\times$3	$\therefore$ T10	

Deductive Argument 10: FR's Alternative Reasoning as a Propositional Calculus with Fallacies Highlighted

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I thank Jose Philip, my friend and mentor who has never failed to point me in the right direction. Both you and Prof. Valerio are the ones who convinced me that I could do my fourth year in NUS. And, I made it... I think. I thank God for His providence that I have a figure like you in my life to make sure I am humble enough to not do anything stupid and confident enough to not do nothing worthwhile. I thank Him that His grace supplied me to complete this race, which I didn't even think I'd be able to start.

I thank my family for being there as a pillar of support and supplier of coffee. I thank you, mom, especially for your encouragement and listening ear when I felt stuck at many parts of this project. I don't say this but I appreciate each breakfast you leave on my desk. Even as I type this now you are out getting dabao caifan and good flask of teh O bing for me. Thanks mom. 10/10

I thank the Bαtαkoαlαs. I won't really miss solving assignments for GR or MM, but I will definitely miss hanging out with you guys in the ~~Physoc Room~~ Computer Lab. And I will certainly miss Mala Mondays. I thank all of you for making uni such a blast (except Jing Hao, I do not thank Jing Hao).

I thank the reader for examining my arguments and I hope you will find that my title is not just "clickbait" or an oversell.

I *hope*.

8 Addendum

Proof that Schmidt Numbers are non-equal and non-zero:

$$|\Psi\rangle_{FR} = \sqrt{\frac{1}{3}}\Big(|tT,\downarrow\Downarrow\rangle + |tT,\uparrow\Uparrow\rangle + |hH,\downarrow\Downarrow\rangle \Big)$$
$$\rho_{FR} = |\Psi\rangle\langle\Psi|_{FR}$$

$$\rho_A = \mathsf{Tr}_B[\rho_{FR}] = \frac{1}{3}\Big(2\,|tT\rangle\langle tT| + |hH\rangle\langle tT| + |tT\rangle\langle hH| + |hH\rangle\langle hH|\Big)$$
$$\det(\rho_A - \lambda_i I) = \left(\frac{2}{3} - \lambda_i\right)\left(\frac{1}{3} - \lambda_i\right) - \frac{1}{9} = 0$$
$$\lambda_i^2 - \lambda + \frac{1}{9} = 0$$
$$\lambda_i = \frac{1 \pm \sqrt{5}/3}{2} = \frac{3 \pm \sqrt{5}}{6}$$

Thusly:

$$\boxed{\lambda_0 = \frac{3+\sqrt{5}}{6}}$$

$$\boxed{\lambda_1 = \frac{3-\sqrt{5}}{6}}$$

Hence λ_0 and λ_1 are non-zero and non-equal.

From here, if one is interested about the eigenvectors, one can slowly go on to find that:

$$|0\rangle_A = \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}}\begin{bmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{bmatrix},\ |1\rangle_A = \frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}}\begin{bmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{bmatrix}.$$

Whereas:

$$|0\rangle_B = \frac{(I \otimes \langle 0|_A)\,|\Psi\rangle_{FR}}{\sqrt{\lambda_0}}$$
$$|1\rangle_B = \frac{(I \otimes \langle 1|_A)\,|\Psi\rangle_{FR}}{\sqrt{\lambda_1}}$$

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