

**National University of Singapore  
Department of Physics**

**PhD Qualifying Comprehensive Examination  
Semester II, AY 2011/2012**

**27 February 2012**

**Paper 2: Long-Answer Questions**

**Time: 2 hours**

**This is a CLOSED-BOOK exam.**

**Answer all FOUR questions.**

**All four questions carry equal weight.**

**Indicate each of your answers CLEARLY.**

### Question 1 — Classical Mechanics

There is a chain of mass  $M$  and length  $L$ .

- (a) Initially the chain is suspended vertically with its lowest end just touching a scale. The chain is released and falls onto the scale. What is the reading of the scale when a length  $x$  of the chain has fallen?
- (b) After the chain has coiled up on the edge of this scale, a very small length at one end is pushed off the edge and starts to fall under the force of gravity, pulling more and more of the chain off the scale. Assume that the velocity of each chain element remains zero until it is jerked into motion with the velocity of the falling section. Find the velocity when a length  $x$  has fallen off.

### Question 2 — Electromagnetism

In a vacuum diode, electrons are “boiled” off a hot cathode, at potential zero, and accelerated across a gap to the anode, which is held at positive potential  $V_0$ . For simplicity, we assume that the cathode and the anode are parallel plates. The cloud of moving electrons within the gap between the plates (called space charge) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on a steady current  $I$  flows between the plates.

Suppose that  $A$ , the area of both plates, is large compared with the separation  $d$  between the plates ( $A \gg d^2$ ). We can then neglect edge effects and regard the potential  $V$ , the charge density  $\rho$ , and the speed  $v$  of the electrons as functions of the distance  $x$  from the cathode.

- (a) Write Poisson’s equation for the region between the plates.
- (b) Assuming that the electrons start from rest at the cathode, what is their speed at point  $x$ , where the potential is  $V(x)$ ?
- (c) In the steady state, current  $I$  is independent of  $x$ . What, then, is the relation between  $\rho(x)$  and  $v(x)$ ?
- (d) Use the results of (a), (b), and (c) to obtain a differential equation for  $V(x)$  upon eliminating  $\rho(x)$  and  $v(x)$ .
- (e) Solve this equation for  $V(x)$  in terms of  $V_0$  and  $d$ .
- (f) Find  $\rho(x)$  and  $v(x)$ .
- (g) Show that  $I = KV_0^{3/2}$  and determine the constant  $K$ .

### Question 3 — Quantum Mechanics

In terms of the creation and annihilation operators, the Hamiltonian of a harmonic oscillator can be written as  $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ .

- (a) Find the eigenstates (eigenkets) of the annihilation operator  $\hat{a}$ . Express your answer in terms of the eigenstates of  $\hat{H}$ .
- (b) Show that if you use an arbitrary eigenstate of  $\hat{a}$  as the initial state, then the system will remain in an eigenstate of  $\hat{a}$  during the time evolution of the harmonic oscillator.
- (c) Can you analogously find the eigenstates of the creation operator  $\hat{a}^\dagger$ ?

### Question 4 — Statistical Physics

Consider noninteracting quantum particles confined in a one-dimensional interval between 0 and  $L$  with the energy eigen spectrum of a *single* particle given by

$$E_n = \frac{\hbar^2\pi^2n^2}{2mL^2} \quad \text{for } n = 1, 2, 3, \dots$$

where  $m$  is the mass. Assuming that the system is in contact with a heat bath of temperature  $T$ , we know that the quantum canonical equilibrium distribution applies.

- (a) If we have *two* identical particles in the interval, and the particles are spinless boson type, give the formula for calculating the partition function  $Z_B$ . There is no need to evaluate your formula to a closed form.
- (b) On the other hand, if the two particles are identical fermions of parallel spins, determine a new formula for the partition function  $Z_F$ .
- (c) Based on your results in (a) and (b), answer if  $Z_B > Z_F$  or  $Z_B < Z_F$ .
- (d) Is the average total energy of the two-particle bosonic system smaller or larger than that of the fermionic system? Justify your answer.

**End of paper**

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Department of Physics**

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Semester I, AY 2012/2013**

**1 October 2012**

**Time: 4 hours**

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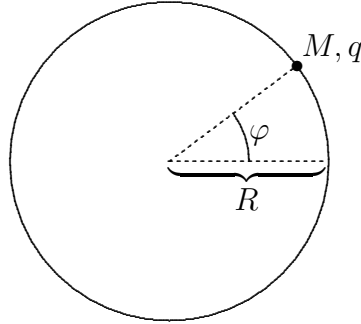
**Each student is allowed to bring in a hand-written cheat sheet of two A4 pages, but no other written or printed material.**

**Answer all FOUR questions.**

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### Question 1 — Quantum Mechanics



A spinless point particle with mass  $M$  and charge  $q$  is moving along a conducting thin wire that has the shape of a circle with radius  $R$ . The wave function of the particle is a function of the azimuth  $\varphi$  and time  $t$ ,  $\psi(\varphi, t)$ . The differential operator for the angular momentum around the axis through the centre of the circle and perpendicular to the plane of the circle is

$$L = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}$$

with  $\hbar = 1.05 \times 10^{-34}$  Js, and

$$H_{\text{kin}} = \frac{L^2}{2MR^2}$$

is the Hamiltonian differential operator for the kinetic energy.

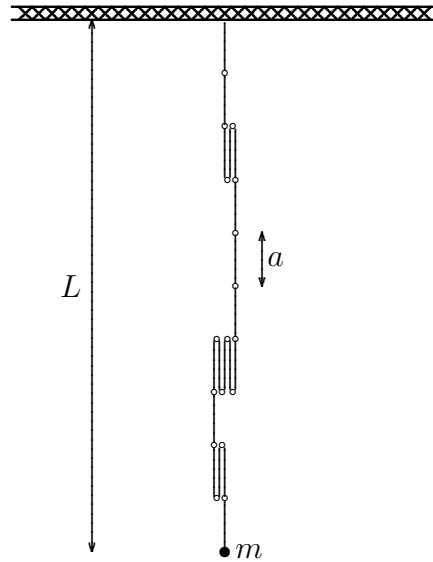
- Explain why the wave function must be periodic,  $\psi(\varphi + 2\pi, t) = \psi(\varphi, t)$ , and state the normalization condition for the wave function.
- Show that the eigenvalues of  $L$  are  $m\hbar$  with  $m = 0, \pm 1, \pm 2, \dots$  and state the normalized eigenfunctions  $u_m(\varphi)$ . What are the corresponding eigenvalues of  $H_{\text{kin}}$ , and what is their degeneracy?
- Use the Schrödinger equation obeyed by  $\psi(\varphi, t)$  and the continuity equation

$$\frac{\partial}{\partial t} \rho(\varphi, t) + \frac{\partial}{\partial \varphi} j(\varphi, t) = 0$$

with the (azimuthal) charge density  $\rho(\varphi, t) = q|\psi(\varphi, t)|^2$  to derive an expression for the current  $j(\varphi, t)$  in terms of  $L$  and  $\psi(\varphi, t)$ .

- For  $M = 9.11 \times 10^{-31}$  kg,  $q = -1.60 \times 10^{-19}$  C, and  $R = 1.2 \times 10^{-6}$  m, how large is the current when the particle is in the eigenstates of  $L$  with  $m = 0$ ,  $m = 1$ , or  $m = 2$ ?

## Question 2 — Statistical Mechanics



One end of a rubber band is fixed to a peg in the ceiling and mass  $m$  is attached to the other end so that the rubber band hangs vertically with length  $L$ . Assume a simple microscopic model of the rubber band as a linked chain of  $N$  segments joined end to end. Each of these segments has the fixed length  $a$  and can orient itself either parallel or antiparallel to the vertical direction. The physical situation has  $N \gg 1$  and  $L \gg a$ . Neglect the mass of the rubber band and any interaction among the segments.

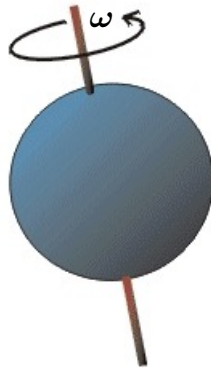
- Find an expression for the energy  $E$  of the system.
- Obtain an expression for the entropy  $S$  of the rubber band in terms of  $L$ ,  $N$ , and  $a$ .
- Find an expression for the Helmholtz free energy  $F$  as a function of the length  $L$  and the temperature  $T$ .
- Minimize  $F$  with respect to  $L$  to obtain the equilibrium length  $L$  of the rubber band as a function of  $m$ ,  $N$ ,  $a$ , and  $T$ .
- If the temperature is increased, does the rubber band get longer or shorter? Justify your answer.

### Question 3 — Electricity and Magnetism

Consider a uniformly charged ball with radius  $R$  and charge density  $\rho$ , with the charge uniformly distributed over the entire volume of the ball. The centre of the ball is at position  $\vec{r} = 0$ .

- Find the electric field strength  $E(r)$  as a function of the distance  $r$  from the centre of the ball. Sketch the plot of  $E$  vs  $r$ .
- What is the relationship between  $E(r)$  and the electrostatic potential  $\Phi(r)$ ? Determine  $\Phi(r)$ , taking the potential at infinity to be zero. Sketch the plot of  $\Phi$  vs  $r$ .
- How much work must be done to bring a point charge  $q$  from far away to the surface of the ball?

Consider now that the uniformly-charged ball rotates with constant angular velocity  $\vec{\omega}$ .



- Show that the electric current density is  $\vec{j} = \rho \vec{\omega} \times \vec{r}$  inside the ball.
- Find the magnetic field at the centre of the ball.

**Question 4 — Mechanics and Thermodynamics**

An isotropic elastic spherical object of radius  $r$  has a tensile surface stress  $\sigma = \partial E / \partial A$ , where  $E$  is the system energy and  $A$  is the surface area;  $\sigma$  has a fixed positive value, so that the surface area tends to shrink. This tendency induces an extra pressure, the so-called Laplace pressure  $P_L$ , on the material inside the surface.

(a) Show that

$$P_L = \frac{2\sigma}{r}$$

when the system is in equilibrium.

- (b) Evaluate  $P_L$  for a spherical aluminum nano-crystal ( $\sigma = 1.25 \text{ N/m}$ ) of radius  $r = 2 \text{ nm}$ , and compare it with the atmospheric pressure of  $10^5 \text{ N/m}^2$ .
- (c) The atomic spacing  $a$  in the nano-crystal decreases in the presence of the Laplace pressure. Use the result of (a) to derive the relative change of the atomic spacing  $\delta a/a$  as a function of  $\sigma$ ,  $r$ , and the so-called bulk modulus  $B_0 = -V (dV/dP)^{-1}$  that quantifies the response of the volume  $V$  to a change of the pressure  $P$ .

**End of paper**



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**4 March 2013**

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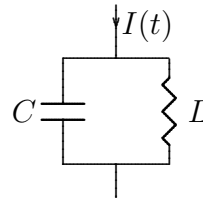
**Question 1 — Rotating rigid body**

A rigid body of mass  $M$  has the shape of an oblate ellipsoid whose surface is given by  $x^2 + y^2 + 4z^2 = 4a^2$  with  $a > 0$ . The body has a homogeneous mass density  $\rho_0$ , except for a ball-shaped core of radius  $a$  that has density  $2\rho_0$ . The body rotates with angular velocity  $\boldsymbol{\omega} = \omega(\mathbf{e}_x \sin \theta + \mathbf{e}_z \cos \theta)$  about an axis through its center.

- (a) Express  $\rho_0$  in terms of  $M$  and  $a$ .
- (b) What is the angular momentum?

## Question 2 — Driven LC circuit

We study the linear electrical circuit described in the figure on the right. For the inductance  $L$  and the capacitance  $C$ , the current in the component is related to the voltages across it by, respectively,



$$V_L = L \frac{dI_L}{dt}, \quad I_C = C \frac{dV_C}{dt}.$$

- (a) Write down Kirchhoff's laws relating  $I_L(t)$ ,  $I_C(t)$ ,  $V_L(t)$ , and  $V_C(t)$ ; then state the equation of motion for

$$\Phi(t) = \int_0^t V(t') dt'.$$

- (b) Find the expressions of  $I_L(t)$  and  $I_C(t)$  when the driving current is  $I(t) = I_0 \cos(\omega t) = I_0 \operatorname{Re}(e^{i\omega t})$ ; then write down the formal expressions of the same quantities in response to a current  $I(t) = I_0 g(t)$ , where  $g(t)$  is a sufficiently well-behaved function of time.
- (c) The dynamics of this electric circuit can also be studied in the Lagrangian formalism. In order to do so, recall that the electromagnetic energy of each element  $u = C, L$  is

$$E_u(t) = \int_0^t I_u(t') V_u(t') dt'.$$

The Lagrangian for this circuit is  $\mathcal{L} = E_C - E_L$  plus a term that accounts for the driving current  $I(t)$ . Express  $\mathcal{L}$  as a functional of  $\{\Phi(t), \dot{\Phi}(t)\}$ , derive the evolution equation for  $\Phi(t)$  and adjust the coupling term to match the equations of motion derived in (a).

- (d) Identify the conjugate momentum  $P$  of  $\Phi$ , then write down the Hamiltonian that governs the dynamics of the circuit.
- (e) In the last decades, experiments have been performed in which the quantum behavior of electric circuits has been demonstrated. Give a formula that estimates the temperature at which the quantization of the energy levels becomes relevant and comment on whether it is easier to observe quantum effects in a circuit with small or with big components. Evaluate your formula for  $L = 1$  nH and  $C = 10$  pF.

The ratio of Planck's and Boltzmann's constants is  $\frac{\hbar}{k_B} = 7.6 \times 10^{-12}$  sK.

### Question 3 — Vibrating string

Consider a continuum model of a one-dimensional string of length  $L$  for which the wave equation

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = 0$$

applies, where  $c$  is the speed of sound. The boundary conditions  $u(0, t) = u(L, t) = 0$  are obeyed at all times  $t$ .

- (a) Determine the normal modes, i.e., the solutions of the form  $u(x, t) = X(x) \cos(\omega t - \varphi)$ . Only a countable number of frequencies,  $\omega_n$ ,  $n = 1, 2, \dots$  are possible. Determine these frequencies and the vibrational modes  $X_n(x)$  associated with them. For the convenience of the following questions, we note that we can normalize  $X(x)$  and different modes are orthogonal, i.e.,  $\int_0^L X_n(x) X_m(x) dx = \delta_{nm}$ , where  $\delta_{nm}$  is the Kronecker delta.
- (b) Write the general solution as a superposition of the normal modes,

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t) X_n(x).$$

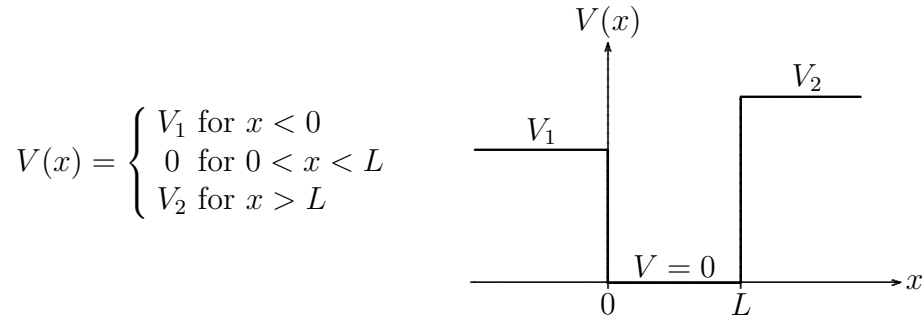
Determine  $q_n(t)$  for the initial conditions

$$u(x, 0) = \left(1 - \left|\frac{2x}{L} - 1\right|\right)a, \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

where  $a$  is a positive constant amplitude.

- (c) In view of this normal-mode decomposition, one can regard the string as a collection of an infinite number of independent harmonic oscillators. State the Hamiltonian  $H$  of these oscillators in terms of the  $q_n(t)$  and their conjugate momenta  $p_n(t)$ . What are the Hamilton's equations of motion for  $q_n(t)$  and  $p_n(t)$ ?
- (d) Compute the partition function  $Z$  for the classical equilibrium canonical distribution at temperature  $T$ .
- (e) Compute the heat capacity  $C$  of the system from the partition function  $Z$ . The result does not make much sense. Explain why.
- (f) Redo the problems of (d) and (e), but consider the system as quantum-mechanical, i.e., the Hamiltonian now is an operator  $\hat{H}$ , and the energy spectrum is now discrete.
- (g) Draw a schematic figure for the heat capacity  $C$  as a function of temperature  $T$ . Determine the limiting forms for large and small  $T$ .

## Question 4 — One-dimensional quantum motion



A particle with mass  $m$  moves in the uneven one-dimensional square potential well shown in the figure above, with  $V_2 > V_1 > 0$ .

- (a) Consider the particle confined in this potential well, that is: the energy  $E$  is such that  $0 < E < V_1$ .
- Sketch the wave functions for the ground state and first excited state. Your wave functions do not have to be exact but the sketches should reflect the essential properties.
  - Write down the general solution in each region.
  - State all matching conditions.
  - Derive an equation from which the energy of the particle can be obtained.
- (b) Consider the particle with energy  $E > V_2$  incident from the left ( $x = -\infty$ ).
- Write down the general solution in each region.
  - State all matching conditions.
  - Explain how the transmission coefficient can be obtained.
- (c) Consider the particle incident from the left ( $x = -\infty$ ), but now the energy of the particle is such that  $V_1 < E < V_2$ . Discuss qualitatively what would happen.

**End of paper**

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**30 September 2013**

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**Question 1 — Stabilization by time-dependent forces**

A charge  $Q$  is placed at each of the four locations:

$$(x, y, z) = (a, 0, 0), (-a, 0, 0), (0, 0, a), \text{ and } (0, 0, -a),$$

which are the corners of a square in the  $xz$  plane.

- (a) Show that near the origin, the electric potential is approximately

$$\phi = \frac{Q}{4\pi\epsilon_0 a} \left[ 4 + \frac{x^2 - 2y^2 + z^2}{a^2} \right].$$

- (b) Show that a particle of mass  $m$  carrying a charge  $q$  placed at the origin will be in equilibrium. Is the origin a stable equilibrium point?
- (c) Let the four charges be rotated about the  $z$ -axis at an angular velocity  $\omega$ . Assuming that  $\omega$  is not too large, and no electromagnetic effects other than the electrostatic field need to be considered, show that the motion of the point charge  $q$  in the rotating frame of reference is governed by the equations:

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - \omega^2 x &= -\frac{Qq}{2\pi\epsilon_0 a^3 m} x, \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y &= \frac{Qq}{\pi\epsilon_0 a^3 m} y, \\ \ddot{z} &= -\frac{Qq}{2\pi\epsilon_0 a^3 m} z. \end{aligned}$$

- (d) Assuming that  $Q$  and  $q$  are of the same sign, show that the origin becomes a stable equilibrium point when the angular velocity is within the range

$$A^2 \leq \omega^2 \leq \frac{9}{8}A^2, \quad \text{where } A^2 = \frac{Qq}{2\pi\epsilon_0 a^3 m}.$$

- (e) Discuss the stability of the origin as an equilibrium point for  $Q$  and  $q$  of opposite sign.

## Question 2 — Point charge in a magnetic field

The classical motion of a charged particle in a constant magnetic field and zero electric field is given by the Lorentz force law

$$m\ddot{\mathbf{x}} = \frac{e}{c}\dot{\mathbf{x}} \times \mathbf{B},$$

where  $m$  is the mass of the particle,  $e$  is its charge,  $\mathbf{x} = (x_1, x_2, x_3)$  is its position vector, and  $\mathbf{B}$  is the magnetic field.

- (a) Describe as quantitatively as you can the motion of such a charged particle if  $\mathbf{B} = (0, 0, B_0)$ . For simplicity, restrict the motion to the  $x_1, x_2$ -plane.
- (b) Consider the Lagrangian

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}),$$

where  $\mathbf{B} = \nabla \times \mathbf{A}$ . Show that the Euler-Lagrange equations give the equation of motion with the Lorentz force that is stated above.

- (c) The canonical momentum  $\mathbf{p} = (p_1, p_2, p_3)$  has components  $p_j = \frac{\partial L}{\partial \dot{x}_j}$ .

Argue why the Hamiltonian can be written as  $H = \frac{1}{2}m\dot{\mathbf{x}}^2$ . Hence show that

$$H = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c}\mathbf{A} \right)^2.$$

- (d) Let  $\mathbf{A} = \frac{1}{2}B_0(-x_2, x_1, 0)$  so that  $\nabla \times \mathbf{A} = (0, 0, B_0)$ . Show that this gives the Hamiltonian

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) - \omega L_3$$

where  $\omega = \frac{eB_0}{2mc}$  and  $L_3 = x_1p_2 - x_2p_1$ .

- (e) Now consider the corresponding quantum-mechanical situation where the components of position  $\mathbf{x}$  and momentum  $\mathbf{p}$  are operators with the usual commutation relations, that is  $[x_j, x_k] = 0$ ,  $[p_j, p_k] = 0$ , and  $[x_j, p_k] = \delta_{jk}i\hbar$ . Show that  $[p_3, H] = 0$  and  $[L_3, H] = 0$  and discuss the implications.
- (e) Find the commutator of the components of the velocity vector  $\mathbf{v} = \frac{1}{m} \left( \mathbf{p} - \frac{e}{c}\mathbf{A} \right)$ .



**Question 3 — A charge near conducting planes**

A charge  $Q$  lies at point  $(x, y, z) = (0, 0, h)$  above an infinite grounded conducting plane at  $z = 0$ .

- (a) Determine the potential and electric field at a point  $\mathbf{r} = (x, y, z)$  above the plane.
- (b) Determine the direction and magnitude of the force on the charge  $Q$  and the work done against the electrostatic forces in assembling this charge.
- (c) Find the surface charge density on the conducting plane.
- (d) Sketch the field lines of the system. If you follow a field line that starts out from the charge  $Q$  in a horizontal direction, that is, parallel to the plane, where does it meet the surface of the conductor?
- (e) If another infinite plane is placed at  $z = h + s$  above the first plane (so the charge  $Q$  is located in between the two planes) and the top plane is connected by a wire to the bottom plane, what is the total surface charge on the surface of *each* plane?

**Question 4 — Lattice of spin-1 atoms**

Consider a lattice with  $N$  spin-1 atoms. Each atom can be in one of the three spin states,  $S_z = -1, 0, +1$ . Let the  $n_{-1}$ ,  $n_0$  and  $n_{+1}$  denote the three respective numbers of atoms in each of these states. Assume that no magnetic field is present, so that all the atoms have the same energy  $E$ .

- (a) Find the total entropy as a function of  $n_{-1}$ ,  $n_0$  and  $n_{+1}$ . Expand any factorials using Stirlings formula.
- (b) Determine the configuration  $(n_{-1}, n_0, n_{+1})$  that maximizes the total entropy.
- (c) Find the total entropy in this configuration.
- (d) Now consider a magnetic field in the  $z$  direction, so that the energy of an atom is  $-g\mu_B B S_z + E$  with the Landé factor  $g$ , the Bohr magneton  $\mu_B$ , and the magnetic inductance  $B$ . Find the partition function for temperature  $T$ . Infer the entropy and confirm that you get the result of (c) in the limit of vanishing field.

**End of paper**