The Geometry of Emparan-Reall Black Rings

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This thesis is submitted in partial fulfilment of the requirements for the degree of Bachelor of Science (Honours) in Physics

Department of Physics
2016
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Acknowledgements

I would like to dedicate this thesis to my parents, Andrew Tan Kian Heng, and Carol Fong Lai Siong for giving me this opportunity to experience undergraduate studies. Thanks to their value of frugality, I am able to pursue my studies without any financial worries. Their countless support and provision of necessities has made me deeply indebted and forever thankful to them. This thesis is also dedicated to my brother, Tan Ren Jun, for his care and concern.

This thesis will also not be made possible without my supervisor Dr. Kenneth Hong Chong Ming. I have been inspired by his passion for physics and had the valuable opportunity to learn under him on numerous occasions. He has been my tutor for PC2131, PC2130, PC2230, lecturer for PC4248 and also my Final Year Project supervisor. It has been a fantastic journey to work with him as he has redefined his role as a supervisor to being also a mentor, educator and friend. Through him, I have learnt technical knowledge such as; general relativity, Maple and gnuplot, and many character development traits such as problem solving. His patience guidance and ability to overlook my numerous shortcomings has made me grateful to him.

I would like to also take this opportunity to also thank my girlfriend, Eunice Lim Li Min for her constant support and prayer when times are tough. Her care and understanding has motivated me on countless occasions and I am really glad that she is by my side during this one year.

The journey was also made much more bearable with my friends, in particular Jeryl Goh, Fredrick Timotius and fellow colleague Cassandra Foo, where they have shared valuable experiences with me and constantly keeping check on me to ensure the completion of this thesis.

I would like to thank my alma mater and all the lecturers that have taught me for an eventful four years of undergraduate studies.
Black ring is essentially a black hole solution that exist in 5 dimensional space-time with a horizon topology of $S^1 \times S^2$. With the discovery of black rings, famous 4 dimensional black hole theorems such as the uniqueness theorem and Hawking’s spherical horizon topology $[1]$, were found to be violated in higher dimensional space-time. This thesis would show the violation and a preliminary analysis of the Emparan-Reall black ring metric with the focus on the geometry of the Emparan-Reall black ring. In general, the Emparan-Reall black ring is unstable, but with the right amount of rotation along the $S^1$ direction, there would be enough centrifugal force to balance the ring’s inward gravitational pull, stabilizing the ring. When we assume the balance condition, the black ring can be parametrized by a scale factor and a shape parameter denoted by $\nu$. The balanced Emparan-Reall black ring can be classified under two families; thin black rings, where $0 < \nu < 1/2$ and fat black rings, where $1/2 < \nu < 1$. In the very thin black ring limit, when $\nu \to 0$, the Emparan-Reall black ring has an almost perfectly spherical $S^2$ event horizon cross-section, while in the very fat black ring limit, when $\nu \to 1$, the $S^2$ event horizon cross-section gets distorted such that it resembles an airfoil shape. The ergosurface, which also has a horizon topology of $S^1 \times S^2$, has an almost perfectly spherical $S^2$ cross-section when $\nu \to 0$, and a highly distorted bulbous shape when $\nu \to 1$. By dropping the balance condition, exotic shapes of the $S^2$ cross-section arises with a “C” shape for the parametric range that does not allow complete embedding and a tear-drop shape for the parametric range which allows complete embedding. This thesis will also show that the thin black ring family are the slowest rotating with the highest spin, while the fat black ring family are the fastest rotating. The extremal black ring which corresponds to a $\nu = 0.5$ black ring has the largest event horizon area but smallest spin.
CHAPTER 1

Introduction

1.1 Brief History

In 1915, Albert Einstein revolutionized our perception of gravity by introducing his theory of general relativity. A few months later, Karl Schwarzschild found the first exact solution that satisfies Einstein’s field equations. This solution was subsequently known as the Schwarzschild metric and it describes the geometry of $D = 4$ space-time surrounding a static, vacuum, massive and spherical object, where $D$ refers to dimensions of space-time. If the spherical object is massive enough, it would curve space-time so much that not even light can escape from it. Such an object is known as a black hole and the boundary beyond which light is unable to escape is known as the event horizon. At the centre of the Schwarzschild black hole lies a point that has infinite mass density; such a point is known as a black hole singularity. It was not until 1963, that an advancement in Einstein’s work was made; Frank Tangherlini managed to generalize the Schwarzschild metric to $D > 4$ space-time and the solution is known as the Schwarzschild-Tangherlini metric [2]. In the same year, another exact solution to Einstein’s field equations was found by Roy Kerr and it is known as the Kerr metric [3]. The Kerr black hole is a Schwarzschild black hole that is rotating. Due to the rotation, the space-time exhibits frame dragging and a new boundary, which envelopes the event horizon, can be defined. This boundary is known as an ergosurface and any object that lies beyond the ergosurface cannot remain stationary with respect to a stationary observer at infinity. In 1986, Robert Myers together with his supervisor, Malcom Perry, managed to generalize the Kerr solution to $D > 4$ space-time and this solution is known as the Myers-Perry (MP) metric [4].

The Schwarzschild, Kerr, and MP solutions all describe space-time surrounding
1.2. MOTIVATIONS FOR HIGHER DIMENSIONS

spherical objects. In those days, it was a common school of thought that all black holes should have a spherical horizon topology. However, in 2002, Roberto Emparan and Harvey S. Reall found a new exact solution to the Einstein’s field equations that describes the geometry of $D = 5$ space-time surrounding a vacuum, stationary object exhibiting a horizon topology of $S^1 \times S^2$ with rotation along the $S^1$ direction. The solution is known as the Emparan-Reall metric \cite{1}. This spurred research into higher dimensions and new exact solutions describing similar objects of horizon topology of $S^1 \times S^2$ were subsequently found. The Figueras metric \cite{5} describes a $D = 5$ space-time surrounding a vacuum, stationary object with rotations along $S^2$ directions while the Pomeransky-Sen’kov metric \cite{6} has rotations along both $S^1$ and $S^2$ directions. This class of solutions are commonly known as black rings (BRs).

1.2 Motivations for higher dimensions

Right now you may start to wonder; why do we need to even study higher dimensional space-time when we live in a $D = 4$ physical universe? This section is dedicated to provide motivations for higher dimension space-time studies.

Firstly, from a theoretical view point, it seems that there is no reason for us to stop at $D = 4$, when Einstein’s theory of relativity allows for physics in higher dimensions. As seen in \cite{7}, classical theories of black holes, such as the ‘no hair’ and uniqueness theorem, developed in 1960’s-70’s, heavily constrained $D = 4$ black holes to be completely and uniquely characterized by just three parameters, angular momentum, mass and charge measured from infinity. Furthermore, all $D = 4$ black holes are known to have a horizon topology of $S^2$. In higher dimensional space-time, such constraining theorems are found not to hold which allows for new exotic shapes like black rings and black saturn, which is a multi black hole solution consisting of a spherical black hole surrounded by a black ring. In addition, black holes in higher dimensions are found to be no longer uniquely specified when given the three parameters.

Secondly, in higher dimensions, there will be extra degrees of freedom allowing gravity to have “richer” dynamics. For stationary, axisymmetric and vacuum solutions, an additional dimension allows for an additional independent plane of rotation.

Thirdly, the Kaluza-Klein reduction allows for compactification of higher dimensions to $D = 4$. This means that the higher dimensions solutions found can be reduced to physical $D = 4$ observable objects. In the case of the Emparan-Reall black ring, a Kaluza-Klein reduction reduces it to a static, charged, spherical $D = 4$ black hole with an additional scalar dilaton field \cite{8}. If such a scalar field is detected, it would be a proof to the existence of higher dimensions space-time and more exciting physics.
Lastly, string theory is currently the most promising candidate to unify all fundamental forces and it requires higher dimensional space-time. In string theory, the fundamental object is a string, and every fundamental particle of nature is being assigned to a vibration mode of the string. This enables string theory to be able to describe both the very large structures and the very small fundamental objects of the universe. In other words, string theory is potentially able to unify the two very contrasting theories of general relativity and quantum mechanics into a single model. Hence, in order to appreciate such a theory, it would be necessary to get ourself familiar with the physics of higher dimensional space-time.

All in all, in the past, it is highly doubted that studies into higher dimensions would be of any physical significances. However, the recent developments as mentioned above provided strong motivations for explorations in this terra incognita.

1.3 Outline of Thesis and some nomenclature

In this thesis, we would define all natural units, \( c = \hbar = G = k_e = k_B = 1 \), and also assign time-like coordinates with the ‘−’ sign while space-like coordinates with a ‘+’ sign. The word horizon comes from the Greek word, horizôn kyklos, which means separating circle. We shall refer the term horizon as a generic boundary separating two different regions of space-time. The two kinds of horizon that we would be interested in this thesis are the event horizon and ergosurface. For simplicity, we would refer all massive objects as black holes from here on.

In Chapter 2, we will review some of the known stationary, axisymmetric and vacuum solutions. In \( D = 4 \) space-time, we only have the Kerr black hole, while in \( D = 5 \) space-time, we will look at the MP black hole. This chapter serves as a good lead-in for us to understand some common properties that stationary, axisymmetric and vacuum solutions have before going to the Emparan-Reall black rings.

Chapter 3 and 4 are the main bulk of this thesis where we will go into the details of the Emparan-Reall metric. In Chapter 3, we will show the violation of uniqueness theorem via a phase diagram, and also in-depth analysis of the Emparan-Reall metric. Physical quantities such as mass, angular momentum, angular speed, and area of event horizon will also be derived.

In Chapter 4, we will use the method of isometric embedding to aid our visualization of the \( S^1 \times S^2 \) horizon topologies of both the event horizon and ergosurface. Various definitions to the radii of the \( S^2 \) and \( S^1 \) event horizon topology will also be introduced in the chapter to aid our understandings of the shape. Due to the technical rather than conceptual nature of the solution, the plots are generated via numerical methods.
In Chapter 5, we will conclude and propose some possible extensions to our current work.
2.1 Introduction

In this chapter, we would review the metric of some known stationary, axisymmetric and vacuum black hole solutions in both the $D = 4$ and $D = 5$ space-time. For $D = 4$, we only have the Kerr metric, whereas, for $D = 5$, we would look at the Myers-Perry (MP) metric. Stationary and axisymmetric solutions are a particular class of solutions that has $D - 2$ commuting Killing vector fields in which one of the Killing vector field has to be $\partial/\partial t$ whereas the term vacuum refers to a manifold in which its Ricci tensor is zero. The vacuum solution is also known as Ricci flat solution, which has a physical connotation that the solution is of pure gravity.

The flow of the chapter will be as such; where we would first look at the metric and derive the locations of some of its pathological features, before extracting some useful physical quantities and subsequently use these extracted physical quantities to plot a phase diagram of the metric.

2.2 Kerr Black Hole

2.2.1 Metric

The Kerr metric describes the geometry of $D = 4$ space-time surrounding a vacuum, stationary, rotating black hole with a spherical horizon of $S^2$. In Boyer-Lindquist coordinates, $(t, r, \theta, \phi)$, the line element of the Kerr metric adapted from [5] is described as such:
\[ ds^2 = -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\Sigma} [a \ dt - (r^2 + a^2) d\phi]^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) \] (2.1)

where
\[ \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2mr + a^2 \]

The parameters \( a \) and \( m \) corresponds to the angular momentum per unit mass and the mass of the black hole respectively. In these coordinates, the metric has two obvious commuting Killing vectors, the time translation Killing vector \( \partial/\partial t \) and the azimuthal Killing vector \( \partial/\partial \phi \). This has a physical connotation which corresponds to the conservation of energy and angular momentum according to Noether’s theorem.

Before we start on anything, it is important to do a quick check if the metric can be reduced to Minkowski space-time. Let us first derive the Minkowski metric in Boyer-Lindquist coordinates. The Boyer-Lindquist coordinates follows the coordinate transformation from the Cartesian coordinates given by,
\[ x = \sqrt{r^2 + a^2 \sin \theta \cos \phi}, \quad y = \sqrt{r^2 + a^2 \sin \theta \sin \phi}, \quad z = r \cos \theta \] (2.2)

Using Minkowski space-time in Cartesian coordinates,
\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \] (2.3)

we can derive the expression of Minkowski space-time in Boyer-Lindquist coordinates and it is given by,
\[ ds^2 = -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \] (2.4)

By letting \( m = 0 \), (2.1) would reduce to the same form as shown above. Also, by letting \( a = 0 \), (2.1) reproduces the Schwarzschild metric in standard Schwarzschild coordinates. By letting \( a = m = 0 \), (2.1) reduces to Minkowski space-time in spherical polar coordinates.

After the quick check, we now look at some of the pathological features of the Kerr metric. We first construct the quadratic curvature invariant given by,
\[ R_{abcd}R^{abcd} = \frac{48m^2 (r^2 - a^2 \cos^2 \theta)[(r^2 + a^2 \cos^2 \theta)^2 - 16r^2 a^2 \cos^2 \theta]}{(r^2 + a^2 \cos^2 \theta)^6} \] (2.5)

From the expression, we can see that \( R_{abcd}R^{abcd} \) blows up at
\[ r = 0, \quad \theta = \pi/2 \] (2.6)

This corresponds to the curvature singularity of the Kerr metric.
The location of the event horizon can be found by calculating the zeroes of $\Delta$, where the component $g_{rr}$ blows up. The zeroes of $\Delta$ are,

$$r_{\pm} = m \pm \sqrt{m^2 - a^2}$$  \hspace{1cm} (2.7)

where $r_+$ is the location of the event horizon while $r_-$ corresponds to the inner horizon. Both horizons exist only under the strict condition that $m \geq |a|$. If $m < |a|$, then both horizons would vanish and Kerr metric would have a naked singularity.

In mathematical terms, the ergosurface is the boundary where the translational time-like Killing vector, $\partial/\partial t$, turns to a null vector. The location can be deduced by solving for the zeroes of $g_{tt}$ and it is found to be at,

$$r_E = m + \sqrt{m^2 - a^2 \cos^2 \theta}$$  \hspace{1cm} (2.8)

in which the other zero given by, $r = m - \sqrt{m^2 - a^2 \cos^2 \theta}$ corresponds to the location of another inner horizon lying beyond the event horizon.

### 2.2.2 Physical Quantities

There are many ways to extract physical quantities from the metric. For this thesis, we shall follow the method used in [9], where we would observe the asymptotic behaviour of the metric and then compare it with the equations found in [9]. A particularly useful feature of the Boyer-Lindquist coordinate is that we can easily deduce the asymptotic behaviour of the metric by taking $r \rightarrow \infty$. The metric takes a simple form at asymptotic infinity given as,

$$ds^2 = - \left[ 1 - \frac{2m}{r} + O(r^{-2}) \right] dt^2 - \left[ \frac{4ma \sin^2 \theta}{r} + O(r^{-2}) \right] d\phi dt + \left[ 1 + \frac{2m}{r} + O(r^{-2}) \right] \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$  \hspace{1cm} (2.9)

The equations found in [9] are stated here,

$$g_{tt} = -1 + \frac{2M}{r} + O(r^{-2}), \quad g_{t\phi} = -2J \frac{\sin^2 \theta}{r} (1 + O(r^{-1})), \quad (2.10)$$

By comparison, we can extract the mass measured at infinity, $M$, and the angular momentum measured at infinity, $J$, which are given by,

$$M = m, \quad J = ma$$  \hspace{1cm} (2.11)

Another physical quantity we can extract from the metric is the area of event horizon, $A_H$. The generic event horizon area formula in $D = 4$ is given as,

$$\text{Area} = \int_0^{2\pi} \int_0^\pi \sqrt{\det(g_{ij})}|_{r=r_+} d\theta d\phi$$  \hspace{1cm} (2.12)
where \((i, j)\) takes values \((\theta, \phi)\). Because we are looking at axisymmetric solutions, the metric tensor is \(\phi\) independent and we can factor out the \(2\pi\). We also note that \(g_{ij}\) is diagonal, hence \(A_H\) can be calculated as such,

\[
A_H = 2\pi \int_0^\pi \sqrt{\left(g_{\theta\theta}g_{\phi\phi}\right)_{r=r_+}} d\theta
\]

\[
= 8m\pi r_+
\]

which can be rearranged to

\[
A_H = 4\pi(r_+^2 + a^2)
\]

and that resembles the surface area of \(S^2\).

### 2.2.3 Phase Diagram

With the physical quantities obtained in the previous subsection, we can now plot the phase diagram of the Kerr metric. The phase diagram can be plotted by using reduced, dimensionless quantities, the reduced area, \(a_H\) and the reduced angular momentum, \(j\) are defined as follows,

\[
a_H = \frac{A_H}{M^2}, \quad j = \frac{J}{M^2}
\]

\[
= \frac{8\pi}{m}(m + \sqrt{m^2 - a^2}), \quad = \frac{a}{m}
\]

The phase diagram, Figure 2.1 is obtained by fixing \(m = 1\) and doing a parametric plot of \(a_H\) against \(j\) over the range of \(0 \leq a \leq m\). We can see that as the angular momentum increase, the area of event horizon decreases. At maximum spin \(j = 1\), the area of event horizon takes the minimum value, \(a_H = 8\pi\).

Figure 2.1: Phase diagram of Kerr metric with \(a_H\) against \(j\)
2.3 Myers-Perry Black Hole

2.3.1 Metric

When $D = 5$, the MP metric describes the geometry of $D = 5$ space-time surrounding a vacuum, rotating black hole with a spherical horizon, $S^3$. In Boyer-Lindquist coordinates, $(t, r, \theta, \psi, \phi)$, the line element of the MP solution adapted from [5] is given as such,

$$ds^2 = -dt^2 + 2m \Sigma (dt - a \sin^2 \theta d\psi - b \cos^2 \theta d\phi)^2$$

$$+ \Sigma (\frac{dr^2}{\Delta} + d\theta^2) + (r^2 + a^2) \sin^2 \theta d\psi^2 + (r^2 + b^2) \cos^2 \theta d\phi^2$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Delta = \frac{(r^2 + a^2)(r^2 + b^2)}{r^2} - 2m$$

The parameters $m$, $a$ and $b$ are corresponds to the mass, and angular momenta per unit mass along the two rotational axes parametrized by the azimuthal angles $\psi$ and $\phi$ respectively. In these coordinates, the $D - 2$ Killing vectors are $\partial/\partial t$, $\partial/\partial \psi$ and $\partial/\partial \phi$, which, like the Kerr solution, corresponds to energy conservations and angular momenta conservation in both independent directions. By letting $a = b = 0$, (2.16) reduces to the $D = 5$ static Tangherlini solution. Similar to the Kerr solution, by letting $m = 0$, we obtain the $D = 5$ Minkowski space-time in Boyer-Lindquist coordinates and by letting $a = b = m = 0$, we obtain the $D = 5$ Minkowski space-time in spherical polar coordinates.

Solving for the zeroes of $\Delta$, we deduce the locations of the event horizon given by,

$$r_{\pm}^2 = \frac{1}{2} \left[ 2m - (a^2 + b^2) \pm \sqrt{[2m - (a + b)^2][2m - (a - b)^2]} \right]$$

(2.17)

where the $r = r_-$ and $r = r_+$ gives the locations of the inner and outer event horizon respectively. It is important to note that both horizons exist and differ upon the condition that $|a| + |b| < \sqrt{2m}$. When $|a| + |b| = \sqrt{2m}$, they coincide and when $|a| + |b| > \sqrt{2m}$ there will be no event horizon and we would have a naked singularity.

Constructing the quadratic curvature invariant, given by,

$$R_{abcd}R^{abcd} = \frac{96m^2(3a^2 \cos^2 \theta - 3b^2 \cos^2 \theta + 3b^2 - r^2)(a^2 \cos^2 \theta - b^2 \cos^2 \theta + b^2 - 3r^2)}{(a^2 \cos^2 \theta - b^2 \cos^2 \theta + b^2 + r^2)^6}$$

(2.18)

we can locate the curvature singularity to be at,

$$r = 0, \quad \cos \theta = \frac{b}{\sqrt{b^2 - a^2}}$$

(2.19)

By solving for the zeroes of $g_{tt}$, we can deduce the location of the ergosurface which is given by,

$$r_E = \sqrt{2m - a^2 \cos^2 \theta - b^2 \sin^2 \theta}$$

(2.20)
2.3.2 Physical Quantities

From (2.16), we are able to extract physical quantities such as the mass, $M$, the angular momenta, $J_\psi$ and $J_\phi$, measured at infinity and the area of event horizon, $A_H$. We first take the limit, $r \to \infty$, and $g_{tt}$, $g_{t\psi}$ and $g_{t\phi}$ takes the asymptotic form given by,

\[ g_{tt} = -1 + \frac{2m}{r^2} + O(r^{-4}), \]
\[ g_{t\psi} = -\frac{2ma \sin^2 \theta}{r^2} + O(r^{-4}), \]
\[ g_{t\phi} = -\frac{2mb \cos^2 \theta}{r^2} + O(r^{-4}). \] (2.21)

We then compare (2.21) with the equations of [9], which the $D = 5$ form of (2.10), and they are listed here as,

\[ g_{tt} = -1 + \frac{8M}{3\pi r^2} + O(r^{-4}), \]
\[ g_{t\psi} = -\frac{4J_\psi \sin^2 \theta}{\pi r^2} (1 + O(r^{-2})), \]
\[ g_{t\phi} = -\frac{4J_\phi \cos^2 \theta}{\pi r^2} (1 + O(r^{-2})) \] (2.22)

and we obtain,

\[ M = \frac{3\pi}{4} m, \quad J_\psi = \frac{\pi}{2} ma, \quad J_\phi = \frac{\pi}{2} mb \] (2.23)

The above shows that parameters $a$, $b$ and $m$ indeed correspond to the angular momenta and mass of the MP black hole. For $A_H$, we set $b = 0$ to simplify calculations and we are only interested in the singly rotating case along the $\psi$ direction. The event horizon area formula in $D = 5$ given by,

\[ A_H = \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \sqrt{\det(g_{ij})} \Big|_{r=r_+} d\theta d\phi d\psi \] (2.24)

where $(i, j)$ take values $(\theta, \psi, \phi)$, and angular coordinates, $\theta$, $\psi$ and $\phi$, have periodicities of $\pi/2$, $2\pi$ and $2\pi$ respectively. Similarly, we can factor out the $2\pi$ from the angular coordinates $\phi$ and $\psi$ and because $g_{ij}$ is diagonal, the area calculation simplifies to,

\[ A_H = 4\pi^2 \int_0^{\pi/2} \sqrt{(g_{\theta\theta}g_{\psi\psi})} \Big|_{r=r_+} d\theta \]
\[ = 4\pi^2 m r_+ a \] (2.25)

where $r_+ = \sqrt{2m - a^2}$, which is $r_+$ when $b = 0$.

2.3.3 Phase Diagram

We can now plot a phase diagram for a singly rotating MP black hole along the $\psi$ direction as shown in Figure [2.22]. The dimensionless reduced area, $a_H$, and reduced angular momentum in the $\psi$ direction, $j_\psi$, can be defined via these relations:

\[ a_H = \frac{3\sqrt{3}}{16\sqrt{\pi}} \frac{A_H}{M^{3/2}}, \quad j_\psi^2 = \frac{27\pi}{32} \frac{J_\psi^2}{M^3} \]
\[ = 2\sqrt{\frac{2m - a^2}{m}} = \frac{a}{\sqrt{2m}} \] (2.26)
where the above definitions are adapted from [10]. The power of $M$ was chosen to make the reduced quantities dimensionless. By setting $m = 1$, we can obtain a parametric plot of $a_H$ against $j_\psi$ over the range of $0 \leq a \leq \sqrt{2}$. Note that $a$ can only take this range of values for $m = 1$ or else $r_\pm$ will become imaginary. This can be seen from (2.17). From Figure 2.2, we can deduce that at minimum spin of $j_\psi = 0$, the area of the event horizon is the largest, where $a_H = 2\sqrt{2}$, while when the MP black hole has maximum spin of $j_\psi = 1$, the area of the event horizon vanishes.

Figure 2.2: Phase Diagram of the singly rotating MP black along the $\psi$ direction hole with $a_H$ against $j_\psi$.
The Study of Emparan-Reall Black Rings

3.1 Introduction

In $D = 4$ space-time, Hawking’s horizon topology theorem states that the spatial of the horizon has to be topologically $S^2$. As research in higher dimensional black holes are commonly based on understandings from their $D = 4$ counterparts, it was a common school of thought back then that black hole theorems which hold true in $D = 4$ space-time should also hold true in higher dimensions. However, in 2002, the first black hole solution, in $D = 5$ space-time, which violates the Hawking’s spherical horizon topology was discovered by Roberto Emparan and Harvey S. Reall. This black hole solution has a spatial horizon cross-section of $S^1 \times S^2$. This can be visualized by attaching a $S^2$ at every point along a circle. Such a visualization gives rise to a ring-like structure, and subsequently, to its name, Black Rings (BR). This chapter is dedicated to the analysis of the Emparan-Reall Black Ring (ERBR).

3.2 Metric

The metric of the ERBR has different forms. For this thesis, we would use the form as adopted in [11 12 13], in which the line element in the toroidal coordinates, $(t, x, y, \psi, \phi)$, is given by,

$$\begin{align*}
ds^2 &= -\frac{F(y)}{F(x)} \left( dt - CR^1 + \frac{y}{F(y)} d\psi \right)^2 \\
& \quad + \frac{R^2}{(x - y)^2} F(x) \left[ -\frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right] \tag{3.1}
\end{align*}$$
3.2. METRIC

where

\[ F(\xi) = 1 + \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 + \nu \xi), \quad C = \sqrt{\lambda(\lambda - \nu)} \frac{1 + \lambda}{1 - \lambda} \]

In this form, the spatial coordinates have ranges defined to be,

\[ -1 \leq x \leq 1, \quad -\infty < y \leq -1, \quad 0 < (\psi, \phi) \leq 2\pi \frac{\sqrt{1 - \lambda}}{1 - \nu} \]

where angular coordinates, \( \psi \) and \( \phi \), are defined to have the above periodicity to ensure regularity at outer infinity located at \( x = y = -1 \). The metric is also parametrized by \( R, \nu \) and \( \lambda \) and they are defined to have ranges given as,

\[ R > 0, \quad 0 < \nu \leq \lambda < 1 \quad (3.2) \]

\( R \) is the scale factor, \( \nu \) is known as the shape parameter, while \( \lambda \) encodes the angular speed. The justifications of the coordinate and parameter ranges will be discussed in subsections 3.2.1, 3.3.3, 3.3.4 and 3.4.1.

The event horizon is found to be located at

\[ y = -1/\nu \quad (3.3) \]

where \( g_{yy} \) blows up. The ergosurface is located at

\[ y = -1/\lambda \quad (3.4) \]

where \( g_{tt} \) changes from time-like to space-like. In order to locate the curvature singularity, we construct the invariant curvature scalar given by,

\[ R_{abcd}R^{abcd} = \frac{3(x - y)^4}{2R^4(x\lambda + 1)^6} \left\{ x^2(3x^4 - 4x^3y + 8x^2y^2 - 2x^2 + 4xy + 3)\lambda^4 \right. \\
+ 2x(7x^4 - 8x^3y + 16x^2y^2 + 2x^2 - 5)\lambda^3 \\
+ 3(9x^4 - 12x^3y + 16x^2y^2 - 6x^2 - 4xy + 1)\lambda^2 \right. \\
+ 8(x^2 - 4xy + 1)(x - y)\lambda + 8(x - y)^2 \nu^2 \\
\left. + \left[ 4x(x^3y - 2x^2 + 3xy + 4)\lambda^4 \right. \\
+ 8(3x^3y + 2x^2 + 3xy - 2)\lambda^3 \\
+ 4(4x^3 + 9x^2y - 10x + 3y)\lambda^2 - 16x(x - y)\lambda \right] \nu \\
\left. - 4(x - 2)(x + 2)\lambda^4 \\
+ 24x\lambda^3 + 4(2x - 1)(2x + 1)\lambda^2 \right\} \quad (3.5) \]

where \( R_{abcd}R^{abcd} \) blows up. Also from \( R_{abcd}R^{abcd} \) we can see that the curvature singularity is located at

\[ y = -\infty \quad (3.6) \]

Similarly, we can see that the asymptotic infinity is located at \( x = y = -1 \) where \( R_{abcd}R^{abcd} \) vanishes.
3.3. ANALYSIS

3.2.1 Ranges of coordinates and parameters

Looking at invariant curvature scalar given in (3.5), a good starting point would be to define the range of \( y \) to be \(-\infty < y \leq -1\), which includes everything from the coordinate singularity to the asymptotic infinity when \( x = -1\). Next, the ranges of the parameters can be easily seen from the expression of \( C \). The \((1 - \lambda)\) term in the denominator of \( C \) restricts \( \lambda < 1 \) and the \( \lambda \) term in the numerator of \( C \) demands \( \lambda \) to be positive-definite. Hence, \( \lambda \) has to take values from \( 0 < \lambda < 1 \). The range of \( \nu \) can be deduced from the \((\lambda - \nu)\) term in the numerator. \( R \) has to be positive-definite as it is the scale factor. The range of \( x \) can be seen from \( g_{\phi\phi} \). In order for \( g_{\phi\phi} \) to remain space-like, we can solve for the zeroes of \( G(x) \) and find them to be \( x = \pm 1 \) and \( x = -1/\nu \). Using the knowledge that \( \nu \) has to lie between \( 0 < \nu \leq \lambda < 1 \), the range of \( x \) has to be chosen as \(-1 \leq x \leq 1\). Since \( g_{\phi\phi} \) vanishes at \( x = \pm 1 \), they corresponds to the poles of the angular coordinate \( \phi \).

3.3 Analysis

In this section, we will go into the details of the analysis of (3.1), where we would show how the \( S^1 \times S^2 \) cross-section can been seen from the metric and other deeper interpretations of the choice of coordinates describing the ERBR.

3.3.1 Toroidal Coordinates

To understand (3.1) better, we shall consider the toroidal coordinates system used. The toroidal coordinates are adapted from the shape of a ring, and this ring shape can be seen by rotating the bipolar coordinate system, as shown in Figure 3.1 along the \( \psi \) direction. Let us first consider the flat space in toroidal coordinates and in polar coordinates, in which their technical derivation is in [7]. The flat spaces are given as,

\[
ds^2 = \frac{R^2}{(x^2 - y^2)} \left[ (y^2 - 1)d\psi^2 + \frac{dy^2}{y^2 - 1} + \frac{dx^2}{1 - x^2} + (1 - x^2)d\phi^2 \right] \]

(3.7)

\[
ds^2 = dr_1^2 + r_1^2d\phi^2 + dr_2^2 + r_2^2d\psi^2 \]

(3.8)

respectively. By setting \( \lambda = \nu = 0 \), we note that spatial part of (3.1) reduces to the flat space metric in (3.7). The latter describes flat space with two radial coordinates and two angular coordinates which allows two independent rotations along the \( \psi \) and \( \phi \) directions. By inspection, the coordinate transformation that brings (3.8) to (3.7) can be seen to be,

\[
r_1 = R \frac{\sqrt{1 - x^2}}{x - y}, \quad r_2 = R \frac{\sqrt{y^2 - 1}}{x - y} \]

(3.9)

From here we can see that the range of \( x \) and \( y \) has to be defined be \(-1 \leq x \leq 1\), and \(-\infty < y \leq -1\), in order for \( r_1 \) and \( r_2 \) to remain real and positive. This coordinate range is coincidentally the same as the one defined in (3.1). The asymptotic infinity
3.3. ANALYSIS

Figure 3.1: Bipolar Coordinates for flat $D = 4$ space with constant $\psi, \phi, \psi + \pi$ and $\phi + \pi$.

is recovered when we take $x \to y \to -1$. The axis of rotation along the $\psi$ direction which is at, $r_2 = 0$, which is a plane instead of a line, is located at $y = -1$. The axis of rotation along the $\phi$ direction which is at $r_1 = 0$, is divided into parts, $x = -1$, which corresponds to outside of the ring, $r_2 \geq R$, and $x = 1$, which corresponds to the inside of the ring, $r_2 \leq R$.

Using (3.9), we can do a parametric plot of $r_1$ against $r_2$ to get the bipolar coordinates as shown in Figure 3.1. Figure 3.1 represents a slice of the flat toroidal coordinates at a constant $\phi, \psi$ and $\psi + \pi, \phi + \pi$. Each dotted circles correspond to spherical surfaces of constant $x$, while the solid circles correspond to spherical surfaces of constant $y$. As $y$ approaches $-\infty$, the spherical surfaces of constant $y$ vanishes to zero size and this corresponds to the location of the ring of radius $R$. As we can see in the following subsections, (3.1) preserves most of the structures of (3.7), just with an added curvature produced by the black ring rotating along the $\psi$ direction.

3.3.2 $S^1$ Topology of the event horizon and ergosurface

From (3.1), it is not obvious that the event horizon of the ERBR has a topology of $S^1 \times S^2$. Let us first identify the $S^1$, and this is done by taking the limit where $y \to -1$. Similar to (3.7), we note that as $y \to -1$, $g_{\psi\psi} \to 0$, hence we can deduce that $y = -1$ corresponds to the axis of rotation for $\psi$. To analysis this, we take a suitable coordinate transformation, where $y = -\cosh[\xi(1 - \nu)/\sqrt{1-\lambda}]$, and take
3.3. ANALYSIS

ξ → 0. The ξψ part of the metric would be given as,

$$ds^2_{\xi \psi} \simeq \frac{R^2(x\lambda + 1)(1 - \nu)}{(1 + x)^2(1 - \lambda)} d\xi^2 + \frac{R^2(x\lambda + 1)(1 - \nu)^3 \xi^2}{(1 + x)^2(1 - \lambda)^2} d\psi^2$$

(3.10)

which is conformal to

$$ds^2_{\xi \psi, \text{con}} = d\xi^2 + \frac{\xi^2(1 - \nu)^2}{1 - \lambda} d\psi^2$$

(3.11)

From here, there are two important features that we need to note. First, the angular coordinate ψ fully parametrize the $S^1$ in ERBR metric. Second, ψ has to be identified with a periodicity of $\Delta\psi = 2\pi \sqrt{1 - \lambda} / (1 - \nu)$ in order for the metric to be regular at the axis, $y = -1$.

3.3.3 $S^2$ Topology of Event Horizon

The $S^2$ topology of the event horizon can be seen by first taking the limit of $y \to -1/\nu$, which corresponds to the location of the event horizon as shown in Section 3.2. The metric describing the event horizon at a constant slice of time is given by,

$$ds^2_{x\phi} \simeq \frac{R^2(1 + \lambda)\lambda(1 - \nu)^2}{\nu(1 - \lambda)(1 + x\lambda)} d\psi^2$$

$$+ \frac{R^2\nu^2}{1 + \nu x} \left[ \frac{1 + x\lambda}{(1 - x^2)(1 + \nu x)^2} d\xi^2 + (1 - x^2)d\phi^2 \right]$$

(3.12)

From the above section, we know that angular coordinate ψ describes $S^1$, so now we look at $x\phi$ part of the metric and it is conformal to,

$$ds^2_{x\phi, \text{con}} = \frac{dx^2}{(1 - x^2)} + \frac{(1 + \nu x)^2}{1 + x\lambda} (1 - x^2) d\phi^2$$

(3.13)

From here, the $S^2$ topology is almost obvious. By taking a suitable coordinate transformation, $x = -\cos \theta$, we would get

$$ds^2_{x\phi, \text{con}} = d\theta^2 + \frac{(1 - \nu \cos \theta)^2}{1 - \lambda \cos \theta} \sin^2 \theta d\phi^2$$

(3.14)

showing the $S^2$ topology of the event horizon. We can further deduce that coordinates $x$ and $\phi$ completely parametrizes the $S^2$ sphere, and that $x$ corresponds to the polar angle $\theta$ in the spherical coordinate system. By taking the limit $x \to +1$ ($\theta \to \pi$) or $x \to -1$ ($\theta \to 0$), we note that $g_{\phi\phi} \to 0$ and hence, like (3.7), these limits correspond to the axis of rotation for $\phi$. The coefficients of $(1 - x^2)d\phi^2$ in (3.13) in the limits is given as,

$$\lim_{x \to \pm 1} \frac{(1 + \nu x)^2}{1 + x\lambda} = \frac{(1 \pm \nu)^2}{1 \pm \lambda}$$

(3.15)

In order for the metric to be regular at either poles, we need to renormalize $\phi$ to $2\pi$. The periodicity of $\phi$ is then demanded to be:

$$\Delta\phi_{x=\pm 1} = 2\pi \sqrt{1 \pm \lambda} / (1 \pm \nu)$$

(3.16)
Looking at the expression, we can see that it is impossible to demand regularity at both poles. As a result, if we demand regularity at the one pole, say at \( x = -1 \), this will incur a conical singularity at the other pole at \( x = +1 \) and vice versa. A conical singularity would correspond to either a deficit angle or an excess angle in the \( \phi \) coordinate. The expression of the deficit angle is given as,

\[
\delta_{x=\pm1} = 2\pi - \frac{1 \mp \nu}{\sqrt{1 \mp \lambda}} \Delta \phi_{x=\pm1} \\
= 2\pi \left[ 1 - \frac{\sqrt{1 \mp \lambda(1 \mp \nu)}}{\sqrt{1 \pm \lambda(1 \pm \nu)}} \right]
\]  

(3.17)

If \( \delta_{x=\pm1} \) gives a negative value, that would correspond to an excess angle.

For the metric in (3.1), we can see that we demanded regularity at \( x = -1 \) as the periodicity of \( \phi \) is given as \( \Delta \phi_{x=-1} = \sqrt{1 - \lambda} \). A plot of the deficit angle, \( \delta_{x=-1} \) and \( \delta_{x=+1} \), against \( \nu \) for different values of \( \lambda \) is shown in Figure 3.2. When we set \( \delta_{x=\pm1} = 0 \), the conical singularity disappears at it would be regular at both poles, causing the ring to be balanced. The balance condition is thus given as,

\[
\lambda_{\text{bal}} = \frac{2\nu}{1 + \nu^2}
\]  

(3.18)

Figure 3.2: \( \delta_{x=+1} \) and \( \delta_{x=-1} \) against \( \nu \) for \( \lambda = 0.005 \) (black), \( \lambda = 0.2 \) (yellow), \( \lambda = 0.5 \) (green), \( \lambda = 0.8 \) (red) and \( \lambda = 0.995 \) (blue)

### 3.3.4 \( S^2 \) Topology of Ergosurface

The same can be done for the ergosurface. By taking the limit \( y \to -1/\lambda \), the \( x\phi \), and applying the coordinate transformation, \( x = -\cos \theta \), the \( \theta\phi \) part of the metric
is given by,
\[ ds^2_{\theta\phi} \simeq R^2 \lambda^2 \left[ \frac{d\theta^2}{(1 - \lambda \cos \theta)(1 - \nu \cos \theta)} + \frac{\sin^2 \theta (1 - \nu \cos \theta) d\phi^2}{(1 - \lambda \cos \theta)^2} \right] \]  
(3.19)

which is conformal to
\[ ds^2_{\theta\phi, con} = d\theta^2 + \frac{(1 - \nu \cos^2 \theta)^2}{1 - \lambda \cos \theta} \sin^2 \theta d\phi^2 \]  
(3.20)

Hence, the topological shape of the ergosurface is also \( S^1 \times S^2 \).

### 3.4 Physical Quantities and Phase Diagram

#### 3.4.1 Coordinate transformation and Physical Quantities

From (3.1), we can extract the mass measured at infinity, \( M \), and the angular momenta measured at infinity, \( J_\psi, J_\phi \). Firstly, we need to cast (3.1) in the spheroidal coordinates \((r, \theta)\). In \( D = 5 \) space-time, the asymptotic Minkowski-space in spheroidal coordinates would be given by,
\[ ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2 + r^2 \sin^2 \theta d\phi^2 \]  
(3.21)

The above metric is adapted from [9]. An ansatz to the coordinate transform would be:
\[ x = -1 + \alpha \frac{\sin^2 \theta}{r^2}, \quad y = -1 - \alpha \frac{\cos^2 \theta}{r^2} \]  
(3.22)

where we choose \( 1/r^2 \) as it corresponds to asymptotic behaviour of \( D = 5 \) flat space-time which we can see it from (2.22). The above choice is also consistent with the limit when \( r \to \infty \), \( x, y \to -1 \). Looking at the scale factor, \( R^2/(x-y)^2 \), in (3.1), we have to assign a different sign in front of the \( \alpha \) or else we would not be able to have the \( 1/r^2 \) behaviour at asymptotic infinity. The assignment of \( \sin^2 \theta \) and \( \cos^2 \theta \) is deduced from (3.21), where \( \sin^2 \theta \) corresponds to \( \phi \) and \( \cos^2 \theta \) corresponds to \( \psi \). Since \( x \) and \( \phi \) parametrizes \( S^2 \) and \( \psi \) parametrizes \( S^1 \), it is intuitive to assign the sinusoidal functions as such.

Applying the coordinate transformation (3.22) and then taking the limit \( r \to \infty \), (3.1) would take the form given by,
\[ ds^2 \simeq - dt^2 \]  
(3.23)

\[ + \frac{2R^2(1 - \lambda)}{\alpha(1 - \nu)} \left[ dr^2 + r^2 d\theta^2 + r^2 \cos^2 \theta \frac{(1 - \nu)^2}{1 - \lambda} d\psi^2 + r^2 \sin^2 \theta \frac{(1 - \nu)^2}{1 - \lambda} d\phi^2 \right] \]

where we expand up to the first-order. It is now obvious that we have to define
\[ \alpha = \frac{2R^2(1 - \lambda)}{1 - \nu} \]  
(3.24)
3.4. PHYSICAL QUANTITIES AND PHASE DIAGRAM

as such and
\[ \tilde{\psi} = \frac{1 - \nu}{\sqrt{1 - \lambda}} \psi, \quad \tilde{\phi} = \frac{1 - \nu}{\sqrt{1 - \lambda}} \phi \] (3.25)
in order to go to (3.21).

Using the above defined coordinate transformation, (3.23) - (3.25), we can take the relevant metric coefficients to the limit \( r \to \infty \) and expand them to the appropriate orders as shown in (2.22). These are given by,
\[ g_{tt} = -1 + 2 \frac{R^2 \lambda}{r^2(1 - \nu)} + O(r^{-4}), \]
\[ g_{t\phi} = 0, \quad g_{t\psi} = -2 \cos^2 \theta \frac{R^3 \sqrt{\lambda(\lambda - \nu)(1 + \lambda)}}{r^2(1 - \nu^2)}(1 + \lambda) + O(r^{-4}) \] (3.26)
The mass and angular momenta measured at infinity are thus identified to be
\[ M = \frac{3\pi R^2 \lambda}{4(1 - \nu)}, \quad J_\phi = 0, \quad J_\psi = \frac{\pi R^3 \sqrt{\lambda(\lambda - \nu)(1 + \lambda)}}{2(1 - \nu^2)} \] (3.27)
We can see that (3.1) indeed describes a \( D = 5 \) black ring rotating along the \( S^1 \) direction. By setting \( \lambda = \nu \), \( J_\psi \) vanishes and (3.1) reduces to the static black ring solution found in [7].

To calculate the area of the event horizon of the ERBR, we first re-normalize the angular coordinates \( \psi \) and \( \phi \) from (3.12) to the regular \( 2\pi \) periodicity. This can be done via (3.25) and the metric now takes the form,
\[ ds^2 \approx \frac{R^2 \nu^2}{1 + \nu x} \left[ \frac{1 + x \lambda}{(1 - x^2)(1 + \nu x)^2} dx^2 + \frac{(1 - x^2)(1 - \lambda)}{(1 - \nu^2)^2} d\phi^2 \right] \] (3.28)
Subsequently, we use (2.25) and the area of event horizon can then be deduced and is given by,
\[ A_H = \Delta \phi \Delta \psi \int_{-1}^{1} \sqrt{\det(g_{ij})}|_{r=-1/\nu} dx \]
\[ = 8\pi^2 R^3 \frac{\sqrt{\nu^3 \lambda(1 - \lambda^2)}}{(1 - \nu^2)(1 + \nu)} \] (3.29)
It will also be interesting to calculate the angular speed of the event horizon denoted by \( \Omega \). We first construct a Killing vector given by,
\[ K^\mu = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \psi} \] (3.30)
By evaluating the norm of \( K^\mu \), at the event horizon and equating it to zero,
\[ g_{\mu \nu} K^\mu K^\nu |_{\psi=-1/\nu} = 0 \] (3.31)
\( \Omega \) can be solved and it is given by,
\[ \Omega = \frac{1}{R} \sqrt{\frac{\lambda - \nu}{\lambda(1 + \lambda)}} \] (3.32)
3.4.2 Phase Diagram and the violation of Uniqueness Theorem

Now we have all of the ingredients needed to show that the black hole uniqueness theorem is indeed violated in $D > 4$ space-time. Using the same definitions in (2.26), and the balance condition in (3.18), we get the following reduced quantities:

$$a_H = 2\sqrt{\nu(1-\nu)}, \quad j_\psi^2 = \frac{(\nu+1)^3}{8\nu} \quad (3.33)$$

By doing a parametric plot over the range of $0 < \nu < 1$ we get Figure 3.3.

![Figure 3.3: Phase Diagram of MP and ERBR showing the violation of Uniqueness Theorem in $D > 4$ space-time](image)

From Figure 3.3 we can see that the ERBR is divided into two families of BRs, the thin BR, for $0 < \nu_{\text{thin}} < 1/2$, and the fat BR, for $1/2 < \nu_{\text{fat}} < 1$. The reason for naming them fat and thin would be clear when we look at Figure 4.7. At $\nu = 1/2$, it gives the extremal BR, in which the BR is neither thin nor fat and is located at the cusp with $j_\psi^2 = 27/32$. The event horizon area of the fat BR vanishes when $j_\psi^2 \to 0$, while the event horizon area of the thin BR vanishes when $j_\psi^2 \to \infty$. The violation of the uniqueness theorem can be seen within the range of, $27/32 \leq j_\psi^2 \leq 1$, in which three different black hole solutions can be seen for a given asymptotic mass, angular momentum and charge, which is zero in our case.

3.4.3 Further Analysis on Physical Quantities

Another useful reduced quantity to look at would be the angular speed which can be defined via,

$$\omega^2 = \frac{8M}{3\pi} \Omega^2 \quad (3.34)$$
Imposing the balance condition (3.18), the reduced angular speed takes the form,

\[ \omega^2 = \frac{2\nu}{1 + 3\nu} \]  

(3.35)

Using (3.33), we construct Table 3.1 and plot Figure 3.4, where in Table 3.1, we keep the values of \( a_H \), \( j^2_{\psi} \) and \( \omega^2 \) rounded off to 3 significant figures. Looking at Table 3.1 and Figure 3.4, we see that at \( \nu = 1/2 \), the extremal BR is corresponds to the largest event horizon area and minimal angular momentum. At both limits where \( \nu \to 0 \), and \( \nu \to 1 \), both areas of the fat and thin BR vanishes. We also note that \( a_H \) is symmetric about \( \nu = 0.5 \). This implies that both fat and thin BRs have the same area when \( \nu_{\text{thin}} = 1 - \nu_{\text{fat}} \). For the fat BRs, \( j^2_{\psi} \to 1 \) and \( \omega \to 1/2 \) as \( \nu \to 1 \), while the thin BRs, \( j^2_{\psi} \to \infty \) and \( \omega \to 0 \) as \( \nu \to 0 \). This implies that the thin BRs family are the slowest rotating BRs with the highest spin, while the fat BRs family are the fastest rotating BRs.

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>0.005</th>
<th>0.02</th>
<th>0.05</th>
<th>0.5</th>
<th>0.8</th>
<th>0.95</th>
<th>0.995</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_H )</td>
<td>0.141</td>
<td>0.280</td>
<td>0.436</td>
<td>1.00</td>
<td>0.800</td>
<td>0.436</td>
<td>0.141</td>
</tr>
<tr>
<td>( j^2_{\psi} )</td>
<td>25.0</td>
<td>6.62</td>
<td>2.90</td>
<td>0.844</td>
<td>0.911</td>
<td>0.975</td>
<td>1.00</td>
</tr>
<tr>
<td>( \omega^2 )</td>
<td>0.0985</td>
<td>0.0377</td>
<td>0.0870</td>
<td>0.400</td>
<td>0.471</td>
<td>0.494</td>
<td>0.499</td>
</tr>
</tbody>
</table>

Table 3.1: Values of \( a_H \), \( j^2_{\psi} \) and \( \omega^2 \) for varying \( \nu \)

Figure 3.4: Plot of \( a_H \) (green), \( j^2_{\psi} \) (blue) and \( \omega^2 \) (brown) against \( \nu \)
4.1 Introduction

In chapter 3 we have seen that a black ring is essentially a black hole solution in $D = 5$ space-time, with horizon topology of $S^1 \times S^2$. In this chapter, we would go in details on how we can better visualize this unusual topology, and how the $S^2$ horizon cross-section would change if we vary the parameters $\lambda$ and $\nu$.

We would first introduce the machinery, known as isometric embedding, that generates all the plots. We would then begin embedding the event horizon and ergosurface of the balanced ERBR. Lastly, we would drop the balance condition, and see how different values of $\lambda$ and $\nu$ would affect the shape of the embedding event horizon.

4.2 Visualizing the $S^2$ horizon cross-section

4.2.1 Isometric Embedding

A method to visualize curved space-time is through isometric embedding. The idea is to map curved 2D surfaces into a non-physical flat 3D Euclidean space, $\mathbb{E}_3$, while preserving the distances at the same time. There are many ways to do isometric embedding, but for the $S^2$ surface of the ERBR, its best if we consider the method used in [14]. Let us begin with a generic curved surface with line element given as,

$$ds^2 = g_{xx} dx^2 + g_{\phi\phi} d\phi^2$$  \hspace{1cm} (4.1)
and we wish to embed it onto a $\mathbb{E}_3$ space given by,

$$ds^2_{\mathbb{E}} = dx_E^2 + dy_E^2 + dz_E^2$$ (4.2)

where we define

$$x_E = f(x) \cos \phi, \quad y_E = f(x) \sin \phi, \quad z_E = g(x)$$ (4.3)

It follows that

$$dx_E^2 = f'(x)^2 \cos^2 \phi dx + f(x)^2 \sin^2 \phi d\phi^2 - 2 f'(x) f(x) \cos \phi \sin \phi dx d\phi$$

$$dy_E^2 = f'(x)^2 \sin^2 \phi dx + f(x)^2 \cos^2 \phi d\phi^2 + 2 f'(x) f(x) \sin \phi \cos \phi dx d\phi$$ (4.4)

$$dz_E^2 = g'(x)^2 dx^2$$

and reduces (4.2) to

$$ds^2_{\mathbb{E}} = [f'(x)^2 + g'(x)^2] dx^2 + f(x)^2 d\phi^2$$ (4.5)

Comparing (4.1) = (4.5), we can see that the functions are given as,

$$f(x) = \sqrt{g_{\phi \phi}}, \quad g(x) = \int_{x_{\text{min}}}^{x'} \sqrt{g_{xx} - f'(x)^2} dx'$$ (4.6)

Looking at (4.6), we can see that in order for a generic curved surface to be isometrically mapped onto $\mathbb{E}_3$, the following condition has to be satisfied

$$g_{xx} \geq \left( \frac{d}{dx} \sqrt{g_{\phi \phi}} \right)^2$$ (4.7)

### 4.2.2 Event Horizon with balance condition

With the machinery developed in subsection 4.2.1, we can isometrically embed the $S^2$ horizon cross-section of the ERBR via numerical integration. The $S^2$ surface is given by the $x\phi$ part of (3.12), and we can re-normalize the angular coordinate $\phi$ to get the $2\pi$ periodicity as shown in (3.28).

By setting $M = 1$, we can fix $R$ and start the embedding process. The expression of $R$ for $M = 1$ is given as such,

$$R = 2 \sqrt{\frac{1 - \nu}{3\pi\lambda}}$$ (4.8)

By substituting $R$, we can simplify the embedding condition given in (4.7) to

$$\nu^2 \lambda - 2\nu + \lambda \geq 0$$

$$\lambda \geq \frac{2\nu}{1 + \nu^2}$$

$$\Rightarrow \lambda_{\text{em}} \geq \lambda_{\text{bal}}, \quad \nu_{\text{em}} \leq \frac{1 - \sqrt{1 - \lambda^2}}{\lambda}$$ (4.9)
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

where $\nu_{em}$ refers to the maximum value that $\nu$ can take for a given value of $\lambda$ for the cross-section to be fully embeddable. From (4.9) and Figure 3.2 we can deduce that any combination of $\lambda$ and $\nu$ which result in an excess angle, i.e. $\delta_{x=-1} < 0$, would automatically imply they cannot be fully embeddable onto the $E_3$ space.

From (4.9), we can see that the balanced $S^2$ cross-section of the ERBR can be fully embedded onto the $E_3$ space. Imposing the balance condition, we plot Figure 9(4.1-4.7) which illustrate the isometrically embedded $S^2$ cross-section of the event horizon for different values of $\nu$.

![Figure 4.1: $S^2$ event horizon cross-section of the balanced ERBR, ($\nu = 0.005$) for a constant $\phi$](image1)

![Figure 4.2: $S^2$ event horizon cross-section of the balanced ERBR, ($\nu = 0.005$)](image2)

![Figure 4.3: $S^2$ event horizon cross-section of the balanced ERBR, ($\nu = 0.5$) for a constant $\phi$](image3)
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

Figure 4.4: $S^2$ event horizon cross-section of the balanced ERBR, ($\nu = 0.5$)

Figure 4.5: $S^2$ event horizon cross-section of the balanced ERBR, ($\nu = 0.995$) for a constant $\phi$

Figure 4.6: $S^2$ event horizon cross-section of the balanced ERBR, ($\nu = 0.995$)

Figure 4.7: $S^2$ event horizon cross-section of the balanced ERBR, for values of $\nu$: [0.005 (black), 0.02 (orange), 0.05 (brown), 0.2 (green), 0.5 (navyblue), 0.8 (purple), 0.95 (red), 0.995 (blue)] for a constant $\phi$
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

We can now see that by increasing the value of $\nu$, the $S^2$ cross-section becomes more distorted away from a $S^2$ shape. For a very thin BR, i.e. $\nu = 0.005$, we can see that the $S^2$ event horizon cross-section is almost perfectly spherical in shape, while for a very fat BR, i.e. $\nu = 0.995$ the $S^2$ event horizon cross-section is elongated and it resembles the shape of an airfoil.

It is important to note that the isometric embedding is done onto a non-physical $E_3$ space, hence it only gives us an idea of how does the shape of the $S^2$ cross-section look like, while preserving distances. Aside from being able to visualize curve $D = 2$ space-time objects, it does not give us any true physical meaning. As a result, the positions of different $S^2$ cross-sections shown in Figure 4.7 and subsequently 4.11 are arbitrary and have no physical meaning.

4.2.3 Ergosurface with balance condition

The same procedure on isometric embedding can be carried out as well for $S^2$ ergosurface cross-section. By taking $y \rightarrow -1/\lambda$ also imposing the balance condition $[3.18]$, and normalizing the angular coordinate $\phi$ to the standard periodicity of $2\pi$, the line element of the $S^2$ cross-section of the ergosurface takes the form,

$$d s^2 = \frac{4\nu^2 R^2}{\nu^2 + 2\nu x + 1} \left[ \frac{d x^2}{(1 + \nu x)(1 - x^2)(1 + \nu^2)} + \frac{(1 + \nu x)(1 - x^2)(\nu^2 - 2\nu + 1)}{(1 - \nu)^2(1 + \nu^2)(\nu^2 + 2\nu x + 1)} d \phi^2 \right]$$

(4.10)

Similar to the event horizon, the $S^2$ cross-section of the balanced ergosurface can be fully embedded onto the $E_3$. Using numerical calculations, we combine the $S^2$ cross-sections of both ergosurface and event horizon for varying values of $\nu$ and they are shown in Figure 4.8 - 4.11. The ergosurface envelops the event horizon and as $\nu$ gets larger, the ergosurface becomes more distorted. Interestingly, for higher values of $\nu$ the event horizon is flatter at the outer ring, while the ergosurface is more curved at the outer ring. From Figure 4.11 we can see that for $\nu = 0.995$, the ergosurface has a very bulbous shape.

![Figure 4.8: Combined $S^2$ ergosurface (blue) and event horizon (purple) cross-sections of the balanced ERBR, ($\nu = 0.005$), for a constant $\phi$](image)
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

Figure 4.9: Combined $S^2$ ergosurface (blue) and event horizon (purple) cross-sections of the balanced ERBR, ($\nu = 0.5$), for a constant $\phi$

Figure 4.10: Combined $S^2$ ergosurface (blue) and event horizon (purple) cross-sections of the balanced ERBR, ($\nu = 0.995$), for a constant $\phi$

Figure 4.11: $S^2$ ergosurface cross-section of the balanced ERBR, for values of $\nu$: (0.005 (black), 0.05 (brown), 0.5(navyblue), 0.8 (purple), 0.95 (red), 0.995 (blue)) for a constant $\phi$

4.2.4 Event Horizon dropping the balance condition

Unlike the balanced case, the $S^2$ event horizon cross-section of the unbalanced ERBR cannot be fully embedded onto $E_3$ in general. Using (4.9), we can calculate $\nu_{\text{em}}$ for
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

Figure 4.12: $S^2$ event horizon cross-sections of the unbalanced ERBR for $\lambda = 0.995$ for a constant $\phi$ for $\nu = 0.995$ (blue), $\nu = 0.8$ (green), $\nu = 0.500$ (navyblue), $\nu = 0.2$ (red) and $\nu = 0.005$ (black).

We shall begin our isometric embedding of the $S^2$ cross-section for a constant $\phi$, by first setting a particular value of $\lambda$ and then plotting for different values of $\nu$. These isometric embeddings can be seen in Figure (4.12 - 4.14). Subsequently, from Figure 4.15, we have the isometric embeddings of the $S^2$ cross-sections which cannot be fully embedded while in Figure 4.16, we have those that can be fully embedded and lastly in Figure 4.17, we will look at the cross-sections of different values of $\lambda$ for $\nu = 0.005$.

Table 4.1: Values of $\nu_{em}$, rounded down to 3 significant figures, for a given value of $\lambda$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.005</th>
<th>0.2</th>
<th>0.5</th>
<th>0.8</th>
<th>0.995</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_{em}$</td>
<td>0.00250</td>
<td>0.101</td>
<td>0.267</td>
<td>0.5</td>
<td>0.904</td>
</tr>
</tbody>
</table>
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

Figure 4.13: $S^2$ event horizon cross-sections of the unbalanced ERBR for $\lambda = 0.8$ for a constant $\phi$ for $\nu = 0.8$ (green), $\nu = 0.5$ (navy blue), $\nu = 0.2$ (red) and $\nu = 0.005$ (black).

The $S^2$ event horizon cross-section of the unbalanced case gives a very different shape as compared to its balanced counterpart. They are more curved at the outer ring unlike the $S^2$ event horizon cross-section of the balanced case, which are more curved at the inner ring, giving rise to a tear-drop shape. For the cross-sections that cannot be fully embedded, they exhibit an abrupt curving near the region in which the embedding gets cut off, giving rise to a “C” shape. There is one aspect of the unbalanced cross-section which are similar to its balanced counterpart which is that a small $\nu$ would corresponds to a small cross-section. Another interesting feature observed is that for a fixed $\nu$, the larger the value of $\lambda$ the more flatten out is the cross-section.

Another interesting feature about the $S^2$ event horizon cross-section of the unbalanced case is that they have a deficit or excess angle in $\phi$. In order to see this, we need to consider a 3D plot of the cross-section by varying $\phi$ as well. In Figure 4.18, we can see a deficit angle of $\delta_{x=-1} \simeq 1.70\pi$, while in Figure 4.19, we can see an excess angle of $\delta_{x=-1} \simeq -0.760\pi$ and the plot is embedded for the range $-1 < x < 0.616$. 
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

Figure 4.14: $S^2$ cross-sections of the ERBR for $\lambda = 0.5$ for a constant $\phi$ for $\nu = 0.5$ (navyblue), $\nu = 0.2$ (red) and $\nu = 0.005$ (black).
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

Figure 4.15: $S^2$ event horizon cross-sections of the unbalanced ERBR for a constant $\phi$ for values of $\lambda = \nu = 0.005$ (black), $\lambda = \nu = 0.2$ (red), $\lambda = \nu = 0.5$ (navyblue), $\lambda = \nu = 0.8$ (green) and $\lambda = \nu = 0.995$ (blue) with their embeddable ranges.

Figure 4.16: $S^2$ event horizon cross-sections of the unbalanced ERBR for a constant $\phi$ which are completely embeddable with values of $\lambda = 0.995$ and $\nu = 0.8$ (blue), $\lambda = 0.995$ and $\nu = 0.5$ (red), $\lambda = 0.995$ and $\nu = 0.2$ (green), $\lambda = 0.8$ and $\nu = 0.2$ (orange), $\lambda = 0.5$, $\nu = 0.2$ (navyblue).
4.2. VISUALIZING THE $S^2$ HORIZON CROSS-SECTION

Figure 4.17: $S^2$ event horizon cross-sections of the unbalanced ERBR for a constant $\phi$ for $\nu = 0.005$ and values of $\lambda = 0.995$ (red), $\lambda = 0.8$ (green), $\lambda = 0.5$ (navyblue) and $\lambda = 0.2$ (orange)

Figure 4.18: $S^2$ cross-section of the ERBR, $(\lambda = 0.995, \nu = 0.5)$ with a deficit angle of $\delta \simeq 1.70\pi$ [not to scale]. The purple portion represents the actual embedded cross-section while the skyblue portion represents the imaginary deficit

Figure 4.19: $S^2$ cross-section of the ERBR, $(\lambda = 0.995, \nu = 0.93)$ with an excess angle of $\delta \simeq -0.760\pi$ [not to scale]. The darkblue portion, which overlaps, represents the excess angle while the skyblue plot represents the cross-section plotted for $0 \leq \phi < 2\pi$
4.3 \( S^2 \) Radius, \( R_2 \) of Event Horizon

Now, we know how does the \( S^2 \) horizon cross-section look like in \( \mathbb{E}_3 \), it would be useful to look at the different definitions of \( S^2 \) radius, \( R_2 \). For the ERBR, the definition of \( R_2 \) is ambiguous due to the distortion. This section will provide a few definitions we can adopt to represent the \( R_2 \). We will follow closely to the definitions adopted in [11]. First, we can define \( R_2 \) at the equator, where the \( S^2 \) is the fattest. Using the normalized event horizon line element \( (3.28) \), together with the balanced condition \( (3.18) \), we can locate the equator by differentiating the metric \( \phi \) component \( g_{\phi\phi} \) with respect to \( x \) and solving it as such,

\[
\frac{d}{dx} \left[ \frac{(1 - x^2)(1 - \lambda)}{(1 - \nu)^2} \right] = 0 \tag{4.11}
\]

we get

\[
x_{eq} = -1 + \sqrt{1 - \nu^2} \tag{4.12}
\]

We reject the other root as it lies outside of the permitted range of \( x \) for \( 0 < \nu < 1 \). Substituting \( x_{eq} \) into metric \( \phi \) component of \( (3.28) \), we arrive at the expression of \( R_2 \) given by,

\[
R_{2, \text{eq}} = R \sqrt{\frac{2 - 2\sqrt{1 - \nu^2}}{1 + \nu^2}} \tag{4.13}
\]

After defining the equator radius, it would also seem natural to define the meridian radius. Recall that the meridian is defined to be the half great circle of constant longitude connecting the poles of \( S^2 \). The meridian of the \( S^2 \) is be found via,

\[
R_{2, \text{mer}} = \frac{1}{\pi} \int_{-1}^{1} \sqrt{g_{xx}} dx
\]

where \( g_{xx} \) is given in \( (3.28) \).

Right now, with \( R_{2, \text{mer}} \) and \( R_{2, \text{eq}} \) we can define a distortion function [12] given by,

\[
\sigma_{\text{distort}} = \frac{R_{2, \text{mer}}}{R_{2, \text{eq}}} - 1 \tag{4.14}
\]

which serves as a measure of distortion. When \( \sigma_{\text{distort}} = 0 \), the \( S^2 \) cross-section is perfectly round. As seen in Figure 4.20 due to gravitation, the \( S^2 \) of ERBR is always distort, but as we approaches the thin ring limit \( j_\psi \to \infty \), \( \sigma_{\text{distort}} \to 0 \). For the fat ring limit, \( j_\psi \to 1 \), we see that \( \sigma_{\text{distort}} \to \infty \).
4.3. $S^2$ RADIUS, $R_2$ OF EVENT HORIZON

Another definition can be deduced from the area of $S^2$ the event horizon, $A_{S^2}$ where $A_{S^2}$ is given by,

$$A_{S^2} = 2\pi \int_{-1}^{1} \sqrt{\det(g_{ij})} dx$$  \hspace{1cm} (4.15)

where $g_{ij}$ is the metric tensor of the event horizon given in, (3.28). We note that in (3.28), the angular coordinate $\phi$ is already normalized in $g_{ij}$ thus, the factor in front of the integral is just $2\pi$ instead of $\Delta \phi$. From (3.28), we can also see that the metric tensor has no cross terms, hence the integration is simplified to,

$$A_{S^2} = 2\pi \int_{-1}^{1} \sqrt{g_{xx}g_{\phi\phi}} dx$$

$$= \frac{8}{3} \frac{1 - \nu}{\sqrt{1 - \nu^2}} \tan^{-1} \left( \frac{\nu}{\sqrt{1 - \nu^2}} \right)$$  \hspace{1cm} (4.16)

Using the formula of the surface area of a sphere, we can define $R_{2, \text{area}}$ as such,

$$R_{2, \text{area}} = \sqrt{\frac{A_{S^2}}{4\pi}}$$ \hspace{1cm} (4.17)

By letting $M = 1$, we can set $R$ and make plots of $R_{2, \text{eq}}$ and $R_{2, \text{area}}$ against $j$ via numerical methods and they are shown in Figure 4.21. It is important to note that the above two definitions are expected to only agree well for the very thin BR family as their horizons are almost spherical in shape and this can be seen in Figure 4.21.
Figure 4.21: $R_{2,\text{eq}}$ (purple), $R_{2,\text{area}}$ (green) and $R_{2,\text{mer}}$ (orange) against $j$ for fixed $M = 1$

In Figure 4.21, the thin BR family of all three definitions of $R_2$ comes from $j = \infty$, and they increase gradually until it reaches the extremal BR at $j = \sqrt{27/32}$. Beyond that point, the curves describe the fat BR family. Both $R_{2,\text{eq}}$ and $R_{2,\text{area}}$ then increase for a short well, before spiraling to zero at $j = 1$, while $R_{2,\text{mer}}$ continues to increase until $j = 1$, where it vanishes. Taking $\nu \to 0$, both definitions of $R_{2,\text{eq}}$ and $R_{2,\text{area}}$ follow $R_2 \to R\nu$, and in the other limit, $\nu \to 1$, they follow $R_2 \to 0$, which corresponds to the fat BR flattening out and soon vanishing at $\nu = 1$.

### 4.4 $S^1$ Radius, $R_1$ of Event Horizon

The $S^1$ is parametrized by angular coordinate $\psi$, and the line element describing the $S^1$ circle of the event horizon can be seen from the metric component $g_{\psi\psi}$ of (3.12). One way to define $R_1$ is by using the area formula of a circle. Such a definition of $R_1$ would take the form of,

$$R_1 = \frac{1}{2\pi} \int_0^{\Delta \psi} \sqrt{\text{det}(g_{\psi\psi})} d\psi$$

$$= \frac{\Delta \psi}{2\pi} \sqrt{g_{\psi\psi}} \quad (4.18)$$

Imposing the balance condition, $R_1$ is given by,

$$R_1 = \frac{R(\nu^2 - 1)}{1 - \nu} \sqrt{\frac{2}{(1 + \nu^2)(\nu^2 + 2\nu x + 1)}} \quad (4.19)$$
For $R_1$, there are a few locations which would be interesting to look at, the inner ring, $x = +1$, the outer ring, $x = -1$ and the equator, $x = x_{eq}$. Their expressions are given below,

$$R_{1, \text{inner}} = R \sqrt{\frac{2}{1 + \nu^2}}, \quad R_{1, \text{outer}} = R \frac{1 + \nu}{1 - \nu} \sqrt{\frac{2}{1 + \nu^2}},$$

$$R_{1, \text{eq}} = R \frac{\sqrt{2} (1 + \nu)}{\sqrt{(1 + \nu^2)(2\sqrt{1-\nu^2} + \nu^2 - 1)}}$$

(4.20)

Via numerical integration, a plot of the above definitions of $R_1$ can be plotted against $j_\psi$ and it is shown below in Figure 4.22 for a fixed $M = 1$.

![Figure 4.22: $R_{1, \text{eq}}$ (purple), $R_{1, \text{inner}}$ (green) and $R_{1, \text{outer}}$ (skyblue) against $j_\psi$ for fixed $M = 1$.](image)

For both $R_{1, \text{eq}}$ and $R_{1, \text{inner}}$, they steadily decrease until they vanish at $j_\psi = 1$. For $\nu \to 0$, all three definitions of $R_1 \to R \sqrt{\lambda/\nu}$. For $\nu \to 1$, both $R_{1, \text{eq}}$ and $R_{1, \text{inner}}$ vanish, while $R_{1, \text{outer}}$ diverges.

Similar to the $R_2$ case, the thin BR class can be seen from $j_\psi = \infty$, gradually decreasing until they reach the extremal BR at $j_\psi = \sqrt{27/32}$. Beyond the extremal point, $R_{1, \text{outer}}$ starts to increase exponentially until it reaches infinity at $j_\psi = 1$. For both $R_{1, \text{eq}}$ and $R_{1, \text{inner}}$, they steadily decrease until they vanish at $j_\psi = 1$. For $\nu \to 0$, all three definitions of $R_1 \to R \sqrt{\lambda/\nu}$. For $\nu \to 1$, both $R_{1, \text{eq}}$ and $R_{1, \text{inner}}$ vanish, while $R_{1, \text{outer}}$ diverges.
4.5 Schematic Diagram of the event horizon of the balanced Emparan-Reall Black Rings

A schematic diagram can be plotted to better illustrate the different definitions of radii mentioned above. This is done via the isometric embedding method introduced in subsection 4.2.1, and hence it is not physical but merely a cartoon to help us visualize the cross-section of the event horizon better.

![Schematic Diagram](image)

Figure 4.23: Schematic Diagram of the event horizon of the balanced ERBR with various radii definitions

Using the knowledge of the above sections, we can summarize by having an artist impression on how the ERBR would look like with azimuthal angular coordinate $\phi$ suppressed. It is to be noted that Figure 4.24 is only schematic visualization of the ring-like shaped horizon of black rings.

![Artist Impression](image)

Figure 4.24: Artist impression of the event horizon of ERBR with azimuthal angular coordinate $\phi$ suppressed
CHAPTER 5

Conclusion and Future work

In summary, this thesis has shown preliminary analysis to understanding stationary, axisymmetric vacuum solutions with main focus on the geometry of the horizons for the ERBR solution. We would like to wrap up this thesis by proposing a few possible extensions to the current work.

5.1 Pomeransky–Sen’kov metric

The Pomeransky-Sen’kov metric is a doubly rotating BR that generalizes the ERBR by allowing rotation along the $S^2$ direction as well. With the isometric embedding defined in subsection 4.2.1, we can carry out similar analysis for the doubly rotating BR case to see how the horizon shape is affected by turning on a rotation along the $S^2$ direction. A similar study has been done by Matthew Carlton in [13], which shows interesting findings that the singly rotating ERBR does not exhibit. One of such is the ergosurface, under the influence of an addition rotation along the $S^2$, merges with itself, resulting in a shift of horizon topology from $S^1 \times S^2$ to $S^3$. Such a process is known as ergoregion merger and a more detailed analysis with regards to this can be found in [14]. Since the $D = 5$ space-time allows for two independent rotations, this area of studying of the Pomeransky-Sen’kov metric would make an interesting topic for future work.

5.2 Adding dipole charge

Another interesting area to look at would be to generalize these vacuum black hole solutions to include charge. Since the BR is a $D = 5$ black hole solution it can also carry a conserved electric charge that satisfy the Einstein-Maxwell-dilaton equations.
However, due to the $S^1 \times S^2$ horizon topology, BRs can carry a new type of magnetic charge known as a “dipole charge”. Such a BR is known as a dipole-charged BR and was first discovered by Emparan as shown in [15], and it too satisfies the Einstein-Maxwell-dilaton equations. A possible area for future work would be to understand how the horizon shape be varied by adding a dipole charge.

5.3 Geodesics and stability of the black rings

A possible extension would be to compute the equations of motion and geodesics along the curved space-time surrounding the BR. Another interesting aspect to work on would be to study the stability of the BR. Knowing all these would enhance our understandings of the BR.


