The Energy-Time Uncertainty Principle

The **objective** of this lecture is to derive the time-energy uncertainty principle.

The energy-time uncertainty principle

\[ \Delta t \Delta E \geq \frac{\hbar}{2} \]

is fundamentally different from position-momentum.

Position, momentum, and energy are all dynamical variables. They are measurable characteristics of the system, at any given time.

On the other hand, time itself is *not* a dynamical variable. Time is the *independent* variable of which the dynamical quantities are *functions*. \( \Delta t \) in the energy-time uncertainty principle is not the standard deviation of a collection of time measurements.

It is the time it takes the system to change substantially.
The Energy-Time Uncertainty Principle

Consider the time derivative of the expectation value of some observable, $Q(x, p, t)$:

$$\frac{d}{dt}\langle Q \rangle = \frac{d}{dt} \langle \Psi | \hat{Q} | \Psi \rangle$$

$$= \langle \frac{\partial \Psi}{\partial t} | \hat{Q} | \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} | \Psi \rangle + \langle \Psi | \hat{Q} \frac{\partial \Psi}{\partial t} \rangle$$

Using the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

$$\frac{d}{dt}\langle Q \rangle = -\frac{1}{i\hbar} \langle \Psi | \hat{H} | \hat{Q} | \Psi \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle + \frac{1}{i\hbar} \langle \Psi | \hat{Q} \hat{H} | \Psi \rangle$$

Since $\hat{H}$ is Hermitian,

$$\langle \langle \Psi | \hat{H} | \hat{Q} | \Psi \rangle = \langle \Psi | \hat{H} \hat{Q} | \Psi \rangle$$

Thus

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$$
The Energy-Time Uncertainty Principle

If \( \hat{Q} \) does not depend explicitly on \( t \),

\[
\frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle
\]

That is, the rate of change of the expectation value is determined by the commutator of the operator with the Hamiltonian.

\( \langle Q \rangle \) would be a constant, or \( Q \) is a conserved quantity, if \( \hat{Q} \) commutes with \( \hat{H} \).

Let \( A = H \) and \( B = Q \) in the generalized uncertainty principle

\[
\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2,
\]

and assume that \( Q \) does not depend explicitly on \( t \), we get

\[
\sigma_H^2 \sigma_Q^2 \geq \left( \frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2 = \left( \frac{1}{2i} i \frac{d\langle Q \rangle}{dt} \right)^2
\]

\[
= \left( \frac{\hbar}{2} \right)^2 \left( \frac{d\langle Q \rangle}{dt} \right)^2
\]
Physical Interpretation of $\Delta t$

The above can be written as

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q\rangle}{dt} \right|$$

Let

$$\Delta E = \sigma_H, \quad \text{and} \quad \Delta t = \frac{\sigma_Q}{|d\langle Q\rangle/dt|}.$$  

Then

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Since

$$\sigma_Q = \left| \frac{d\langle Q\rangle}{dt} \right| \Delta t,$$

$\Delta t$ represents the amount of time it takes the expectation value of $Q$ to change by one standard deviation.

$\Delta t$ depends entirely on what observable ($Q$). The change can be rapid for one observable and slow of another. But if $\Delta E$ is small, then the rate of change of all observables must be very gradual, and conversely, if any observable changes rapidly, the “uncertainty” in the energy must be large.
Example: Stationary State

In the extreme case of a stationary state, the energy is uniquely determined, $\Delta E = 0$. All expectation values are constant in time, $\Delta t = \infty$.

To make something happen, you must take a linear combination of at least two stationary states, for example,

$$
\Psi(x, t) = a\psi_1(x)e^{-iE_1t/\hbar} + b\psi_2(x)e^{-iE_2t/\hbar}.
$$

If $a$, $b$, $\psi_1$, and $\psi_2$ are real,

$$
|\Psi(x, t)|^2 = a^2[\psi_1(x)]^2 + b^2[\psi_2(x)]^2 + 2ab\psi_1(x)\psi_2(x)\cos \left( \frac{E_2 - E_1}{\hbar}t \right)
$$

The period of oscillation is $\tau = 2\pi\hbar/(E_2 - E_1)$. Roughly, then, $\Delta E = E_2 - E_1$ and $\Delta t = \tau$, so

$$
\Delta E \Delta t = 2\pi\hbar > \frac{\hbar}{2}
$$
Application of Uncertainty Principles

The $\Delta$ particle lasts about $10^{-23}$ seconds before spontaneously disintegrating. If you make a histogram of all measurements of its mass, you get a kind of bell-shaped curve centered at $1232$ MeV/$c^2$, with a width of about $115$ MeV/$c^2$. Why does the rest energy ($mc^2$) sometimes come out higher than 1232, and sometimes lower?

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \]

For

\[ \Delta t \approx 10^{-23} \text{s} \]

\[ \Delta E \geq \frac{\hbar}{2 \Delta t} \approx 0.527 \times 10^{-11} \text{ J} \approx 33 \text{ MeV} \]

\[ E = mc^2 \]

\[ \Delta m = \Delta E / c^2 \geq 33 \text{ MeV}/c^2 \]
Application of Uncertainty Principles

Use the uncertainty relation to estimate the ground state energy of a harmonic oscillator.

The energy is given by

\[ E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \]

For a harmonic oscillator,

\[ \langle x \rangle = 0, \quad \langle p \rangle = 0 \]

\[ (\Delta x)^2 = \sigma_x^2 = \langle x^2 \rangle \]

\[ (\Delta p)^2 = \sigma_p^2 = \langle p^2 \rangle \]

\[ \langle H \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{(\Delta p)^2}{2m} + \frac{1}{2} m\omega^2 (\Delta x)^2 \]

Using

\[ a + b \geq 2\sqrt{ab} \]

\[ \langle H \rangle \geq \sqrt{(\Delta p)^2 (\Delta x)^2} \omega = \Delta x \Delta p \omega \geq \frac{1}{2} \hbar \omega \]

\[ \langle H \rangle_{\text{min}} = \frac{1}{2} \hbar \omega \]