Hilbert Space

To complete the space, we throw in all square-integrable convergent sequences of functions in the space.

This does not require the introduction of any new basis, we only need to allow linear combinations involving an infinite number of terms.

\[ |\alpha\rangle = \sum_{j=1}^{\infty} a_j |e_j\rangle, \]

provided \( \langle \alpha | \alpha \rangle \) is finite – which is to say (if the basis is orthonormal), provided

\[ \sum_{j=1}^{\infty} |a_j|^2 < \infty. \]

A complete inner product space is called a Hilbert space.
Hilbert Space

The the \( P(\infty) \) space can be completed by including all square-integrable functions on the interval \(-1 < x < 1\) in the set. It is denoted by \( L_2(-1, 1) \).

In principles, the set of all polynomials, \( P(\infty) \), includes functions of the form

\[
f_N(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^N}{N!}
\]

(for finite \( N \)), but it does not include the limit as \( N \to \infty \):

\[
1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x
\]

This is because \( e^x \) is not itself a polynomial, although it is the limit of a sequence of polynomials. To complete the space, we include all such functions.

More generally, the set of all square-integrable functions on the interval \( a < x < b \) is \( L_2(a, b) \).

Quantum mechanical wave functions live in the Hilbert space \( L_2(-\infty, \infty) \) (or \( L_2 \) for short).
Spectrum

The eigenfunctions of the Hermitian operators $i\hat{D} = id/dx$ and $\hat{x} = x$ are of particular importance. They take the form

$$f_\lambda(x) = A_\lambda e^{-i\lambda x}, \quad \text{and} \quad g_\lambda(x) = B_\lambda \delta(x - \lambda)$$

respectively.

Their eigenvalues can be any real number.

The set of all eigenvalues of a given operator is called its spectrum.

$i\hat{D}$ and $\hat{x}$ are operators with continuous spectra, in contrast to the discrete spectra we have seen before (eigenvalues of the Hamiltonian operator for 1D harmonic oscillator, or 1D square well potential, or hydrogen atom).
Eigenfunctions of $i\hat{D}$ and $\hat{x}$

Strictly speaking, the wavefunctions

\[ f_\lambda(x) = A_\lambda e^{-i\lambda x}, \quad \text{and} \quad g_\lambda(x) = B_\lambda \delta(x - \lambda) \]

do not lie in Hilbert space and they cannot be countered as vectors in the Hilbert space, because they are not square-integrable.

\[
\int_{-\infty}^{\infty} f_\lambda(x)^* f_\lambda(x) \, dx \\
= |A_\lambda|^2 \int_{-\infty}^{\infty} e^{i\lambda x} e^{-i\lambda x} \, dx \\
= |A_\lambda|^2 \int_{-\infty}^{\infty} 1 \, dx \to \infty
\]

\[
\int_{-\infty}^{\infty} g_\lambda(x)^* g_\lambda(x) \, dx \\
= |B_\lambda|^2 \int_{-\infty}^{\infty} \delta(x - \lambda) \delta(x - \lambda) \, dx \\
= |B_\lambda|^2 \delta(x - \lambda) \to \infty
\]
Eigenfunctions of $i\hat{D}$ and $\hat{x}$

Anyway, they do satisfy a kind of orthogonality condition:

$$
\int_{-\infty}^{\infty} f_\lambda(x)^* f_\mu(x) \, dx = A_\lambda^* A_\mu \int_{-\infty}^{\infty} e^{i\lambda x} e^{-i\mu x} \, dx
$$

$$
= |A_\lambda|^2 2\pi \delta(\lambda - \mu)
$$

$$
\int_{-\infty}^{\infty} g_\lambda(x)^* g_\mu(x) \, dx = B_\lambda^* B_\mu \int_{-\infty}^{\infty} \delta(x - \lambda) \delta(x - \mu) \, dx
$$

$$
= |B_\lambda|^2 \delta(\lambda - \mu)
$$

The following are treated as the “normalized” eigenfunctions of $i\hat{D}$ and $\hat{x}$, respectively:

$$
f_\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{-i\lambda x}, \quad \text{with} \quad \langle f_\lambda | f_\mu \rangle = \delta(\lambda - \mu)
$$

$$
g_\lambda(x) = \delta(x - \lambda), \quad \text{with} \quad \langle g_\lambda | g_\mu \rangle = \delta(\lambda - \mu)
$$
Eigenfunctions of $i\hat{D}$ and $\hat{x}$

Using the “normalized” eigenfunctions of $i\hat{D}$ and $\hat{x}$ as bases for $L_2$, the linear combination becomes an integral:

$$|f\rangle = \int_{-\infty}^{\infty} a_\lambda |f_\lambda\rangle d\lambda,$$

$$|f\rangle = \int_{-\infty}^{\infty} b_\lambda |g_\lambda\rangle d\lambda.$$ 

Taking the inner product with $|f_\mu\rangle$, and exploiting the “orthonormality” of the basis, we can obtain the “components” $a_\lambda$:

$$\langle f_\mu | f\rangle = \int_{-\infty}^{\infty} a_\lambda \langle f_\mu | f_\lambda\rangle d\lambda = \int_{-\infty}^{\infty} a_\lambda \delta(\mu - \lambda) = a_\mu$$

So

$$a_\lambda = \langle f_\lambda | f\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx = \bar{f}(-\lambda)$$

That is, the $\lambda$ “component” of the vector $|f\rangle$, in the basis of eigenfunctions of $i\hat{D}$, is the Fourier transform of the function $f(x)$. 

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Likewise
\[ \langle g_\mu | f \rangle = \int_{-\infty}^{\infty} b_\lambda \langle g_\mu | g_\lambda \rangle d\lambda = \int_{-\infty}^{\infty} b_\lambda \delta(\mu - \lambda) = b_\mu. \]

\[ b_\lambda = \langle g_\lambda | f \rangle = \int_{-\infty}^{\infty} \delta(x - \lambda) f(x) dx = f(\lambda) \]

That is, the \( \lambda \) “component” of the vector \( | f \rangle \) in the position basis is \( f(\lambda) \) itself.

It is important to understand that \( | f \rangle \) is a abstract vector, and it can be expressed with respect to any basis you like. In this sense, the function \( f(x) \) is merely the collection of its “components” in the particular basis consisting of eigenvectors of the position operator.

In these bases, it becomes difficult to represent operators by matrices because the basis vectors are labeled by a nondenumerable index. Nevertheless, we are still interested in quantities of the form

\[ \langle f_\lambda | \hat{T} | f_\mu \rangle \]

They are still referred as the \( \lambda, \mu \) matrix element—of the operator \( \hat{T} \).