Eigenvectors and Eigenvalues

In a complex vector space, every linear transformation has “special” vectors which are transformed into simple multiples of themselves:

$$\hat{T}|\alpha\rangle = \lambda|\alpha\rangle$$

Such vectors are called eigenvectors of the transformation, and the (complex) number $\lambda$ is their eigenvalue. (The null vector does not count!)

**Examples:**

Vectors that lie along the axis of rotation.

$$\hat{T}|\alpha\rangle = |\alpha\rangle$$

Vectors lie in the “equatorial” plane for rotation by $180^\circ$.

$$\hat{T}|\alpha\rangle = -|\alpha\rangle$$
Matrix Representation

With respect to a particular basis, the eigenvector equation assumes the matrix form

\[ Ta = \lambda a, \quad \text{or} \quad (T - \lambda I)a = 0 \]

For non-zero \( a \), the determinant of \( (T - \lambda I) \) must vanish.

\[ \det(T - \lambda I) = \begin{vmatrix} (T_{11} - \lambda) & T_{12} & \cdots & T_{1n} \\ T_{21} & (T_{22} - \lambda) & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & (T_{nn} - \lambda) \end{vmatrix} = 0 \]

Expansion of the determinant yields an algebraic equation for \( \lambda \):

\[ C_n\lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0 = 0 \]

where the coefficients \( C_i \) depend on the elements of \( T \). This is called the characteristic equation for the matrix; its solutions determine the eigenvalues. The eigenvectors can be constructed by plugging each \( \lambda \) into the matrix equation.
Diagonal Form

If the eigenvectors span the space, we are free to use them as a basis:

\[
\hat{T}|f_1\rangle = \lambda_1|f_1\rangle \\
\hat{T}|f_2\rangle = \lambda_2|f_2\rangle \\
\vdots \quad \vdots \\
\hat{T}|f_n\rangle = \lambda_n|f_n\rangle
\]

The matrix representing \( \hat{T} \) takes on a very simple form in this basis, with the eigenvalues strung out along the main diagonal and all other elements zero:

\[
T = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\]
Diagonal Form

The (normalized) eigenvectors are equally simple:

\[
\mathbf{a}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{a}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \mathbf{a}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

A matrix that can be brought to diagonal form by a change of basis is said to be diagonalizable. The similarity matrix that accomplishes the transformation can be constructed by using the eigenvectors (in the old basis) as the columns of \( \mathbf{S}^{-1} \).

\[
(\mathbf{S}^{-1})_{ij} = (\mathbf{a}^{(j)})_i
\]

There is a great advantage in bringing a matrix to diagonal form – it’s much easier to work with.

Unfortunately, not every matrix can be diagonalized – the eigenvectors have to span the space.
Hermitian Conjugate of $\hat{T}$

Hermitian conjugate of a matrix

$$T^\dagger = \hat{T}^*$$

A more fundamental definition for Hermitian conjugate of a linear transformation:

$$\langle \alpha | (\hat{T} | \beta) \rangle = (\langle \alpha | \hat{T}^\dagger \rangle | \beta)$$

Here $\langle \alpha | \hat{T}^\dagger = (\hat{T} | \alpha) \rangle^\dagger$. This means when the transformation $\hat{T}^\dagger$ is applied to the first member of an inner product, gives the same result as if $\hat{T}$ itself had been applied to the second vector.

$$\langle \alpha | (c | \beta) \rangle = c \langle \alpha | \beta \rangle, \quad \text{but} \quad (\langle \alpha | c \rangle | \beta) = c^* \langle \alpha | \beta \rangle$$

for any scalar $c$.

Using orthonormal basis, the Hermitian conjugate of a linear transformation is represented by the Hermitian conjugate of the corresponding matrix

$$\langle \alpha | (\hat{T} | \beta) \rangle = a^\dagger T b = (T^\dagger a)^\dagger b = (\langle \alpha | \hat{T}^\dagger \rangle | \beta)$$
Hermitian Transformation

In quantum mechanics, a fundamental role is played by Hermitian transformations \((\hat{T}^\dagger = \hat{T})\). The eigenvectors and eigenvalues of a Hermitian transformation have three crucial properties:

1. **The eigenvalues of a Hermitian transformation are real.**

   Let \(\lambda\) be an eigenvalue of \(\hat{T}\): \(\hat{T}|\alpha\rangle = \lambda|\alpha\rangle\), with \(|\alpha\rangle \neq 0\). Then

   \[
   \langle \alpha | (\hat{T}|\alpha\rangle) = \langle \alpha | (\lambda|\alpha\rangle) = \lambda \langle \alpha |\alpha\rangle.
   \]

   On the other hand, if \(\hat{T}\) is Hermitian, then

   \[
   \langle \alpha | (\hat{T}|\alpha\rangle) = (\langle \alpha |\hat{T}^\dagger \rangle |\alpha\rangle = (\langle \alpha |\hat{T} \rangle |\alpha\rangle
   \]

   \[
   = (\langle \alpha |\lambda \rangle |\alpha\rangle = \lambda^* \langle \alpha |\alpha\rangle
   \]

   But \(\langle \alpha |\alpha\rangle \neq 0\), so \(\lambda = \lambda^*\), and hence \(\lambda\) is real.
Hermitian Transformation

2. The eigenvectors of a Hermitian transformation belonging to distinct eigenvalues are orthogonal.

Suppose $\hat{T}|\alpha\rangle = \lambda|\alpha\rangle$ and $\hat{T}|\beta\rangle = \mu|\beta\rangle$, with $\lambda \neq \mu$. Then

$$\langle \alpha|(\hat{T}|\beta\rangle) = \langle \alpha|\mu|\beta\rangle = \mu\langle \alpha|\beta\rangle,$$

On the other hand, if $\hat{T}$ is Hermitian

$$\langle \alpha|(\hat{T}|\beta\rangle) = (\langle \alpha|\hat{T}^\dagger\rangle)|\beta\rangle = \lambda^*\langle \alpha|\beta\rangle.$$

Note that $\lambda^* = \lambda$ (from property 1), and $\lambda \neq \mu$, by assumption, so

$$\langle \alpha|\beta\rangle = 0$$

i.e. the two eigenvectors are orthogonal.
Hermitian Transformation

3. The eigenvectors of a Hermitian transformation span the space (form a complete set of basis).

If all \( n \) roots of the characteristic equation are distinct, then (by property 2) we have \( n \) mutually orthogonal eigenvectors, so they obviously span the space.

If there are duplicate roots (degenerate eigenvalues), it can be shown that there are \( m \) linearly independent eigenvectors with the same eigenvalue if the eigenvalue is \( m \)-fold degenerate.

These eigenvectors can be orthogonalized by the Gram-Schmidt procedure.

Any Hermitian matrix can be diagonalized by a similarity transformation, with \( S \) unitary. This is the mathematical support on which much of quantum mechanics leans.