Linear Transformations

Familiar examples of linear transformations:

- Multiply every vector in 3D-space by 17;
- Rotate every vector by 39° about the z-axis;
- Reflect every vector in the \( xy \)-plane.

In general, a **linear transformation** \( \hat{T} \) takes each vector in a vector space and “transform” it into some other vector \( (|\alpha\rangle \rightarrow \hat{T}|\alpha\rangle) \), on condition that the operation is **linear**:

\[
\hat{T}(a|\alpha\rangle + b|\beta\rangle) = a(\hat{T}|\alpha\rangle) + b(\hat{T}|\beta\rangle)
\]

for any vectors \( |\alpha\rangle, |\beta\rangle \) and any scalars \( a, b \).

If we know what a particular linear transformation does to a set of *basis* vectors, then we can easily figure out what it does to *any* vector.

Suppose

\[
\hat{T}|e_1\rangle = T_{11}|e_1\rangle + T_{21}|e_2\rangle + \cdots + T_{n1}|e_n\rangle
\]
\[
\hat{T}|e_2\rangle = T_{12}|e_1\rangle + T_{22}|e_2\rangle + \cdots + T_{n2}|e_n\rangle
\]
\[ \hat{T} |e_n\rangle = T_{1n} |e_1\rangle + T_{2n} |e_2\rangle + \cdots + T_{nn} |e_n\rangle \]
or, more compactly,
\[ \hat{T} |e_j\rangle = \sum_{i=1}^{n} T_{ij} |e_i\rangle, \quad (j = 1, 2, \cdots, n) \]

If \( |\alpha\rangle \) is an arbitrary vector:
\[ |\alpha\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle + \cdots + a_n |e_n\rangle = \sum_{j=1}^{n} a_j |e_j\rangle, \]

then
\[ \hat{T} |\alpha\rangle = \sum_{j=1}^{n} a_j (\hat{T} |e_j\rangle) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_j T_{ij} |e_i\rangle = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} T_{ij} a_j \right) |e_i\rangle = \sum_{i=1}^{n} a'_{i} |e_i\rangle \]
i.e., \( \hat{T} \) takes a vector with components \( a_1, a_2, \cdots, a_n \) into a vector with components
\[ a' = \sum_{j=1}^{n} T_{ij} a_j \]
Matrix Representation

The $n^2$ elements $T_{i,j}$ uniquely characterize the linear transformation $\hat{T}$ (with respect to a given basis).

If the basis is orthonormal, multiply the equation

$$\hat{T}|e_j\rangle = \sum_{i=1}^{n} T_{i,j}|e_i\rangle$$

by $\langle e_k|$ from the left

$$\langle e_k|\hat{T}|e_j\rangle = \sum_{i=1}^{n} T_{i,j}\langle e_k|e_i\rangle = \sum_{i=1}^{n} T_{i,j}\delta_{i,k} = T_{k,j}$$

That is

$$T_{i,j} = \langle e_i|\hat{T}|e_j\rangle$$

It is convenient to write this in matrix form

$$T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{pmatrix}$$

The study of linear transformation, then, reduces to the theory of matrices.
Matrix Operations

The **sum** of two linear transformations \((\hat{S} + \hat{T})\) is defined in the natural way:

\[(\hat{S} + \hat{T})|\alpha\rangle = \hat{S}|\alpha\rangle + \hat{T}|\alpha\rangle.\]

This matches the usual rule for adding matrices (adding the corresponding elements)

\[U = S + T \iff U_{ij} = S_{ij} + T_{ij}\]

The **product** of two linear transformations \((\hat{S}\hat{T})\) is the net effect of performing them in succession – first \(\hat{T}\), then \(\hat{S}\):

\[|\alpha\rangle \rightarrow |\alpha'\rangle = \hat{T}|\alpha\rangle \rightarrow |\alpha''\rangle = \hat{S}|\alpha'\rangle = \hat{S}(\hat{T}|\alpha\rangle) = \hat{S}\hat{T}|\alpha\rangle\]

Let \(\hat{U} = \hat{S}\hat{T}\), then

\[a''_i = \sum_{j=1}^{n} S_{ij}a'_j = \sum_{j=1}^{n} S_{ij} \left( \sum_{k=1}^{n} S_{jk}a_k \right) = \sum_{k=1}^{n} \left( \sum_{j=1}^{n} S_{ij}T_{jk} \right) a_k = \sum_{k=1}^{n} U_{ik}a_k.\]
Therefore

\[ \mathbf{U} = \mathbf{S} \mathbf{T} \quad \Leftrightarrow \quad U_{ik} = \sum_{j=1}^{n} S_{ij} T_{jk} \]

This is the standard rule for matrix multiplication – to find the \( ik \)th element of the product, we look at the \( i \)th row of \( \mathbf{S} \) and the \( k \)th column of \( \mathbf{T} \), multiply corresponding entries, and add.

The same procedure applies to multiplication of rectangular matrices, as long as the number of columns in the first matches the number of rows in the second.

In particular, if we write the \( n \)-tuple of components of \( |\alpha\rangle \) as an \( n \times 1 \) column matrix

\[
\mathbf{a} = \begin{pmatrix}
  a_1 \\
  a \\
  \vdots \\
  a_n
\end{pmatrix}
\]

the transformation rule can be written

\[ \mathbf{a}' = \mathbf{T} \mathbf{a}. \]
Transpose of a Matrix: $\tilde{T}$

The **transpose** of a matrix is the same set of elements, but with rows and columns interchanged:

$$\tilde{T} = \begin{pmatrix} T_{11} & T_{21} & \cdots & T_{n1} \\ T_{12} & T_{22} & \cdots & T_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix}$$

The transpose of a column matrix is a **row matrix**

$$\tilde{a} = (a_1\ a_2\ \cdots\ a_n)$$

A square matrix is **symmetric** if it is equal to its transpose (reflection in the **main diagonal** – upper left to lower right – leaves it unchanged); it is **antisymmetric** if this operation reverses the sign:

**Symmetric:** $\tilde{T} = T$

**Antisymmetric:** $\tilde{T} = -T$
Complex Conjugate of a Matrix: $T^*$

To construct the (complex) **conjugate** of a matrix, we take the complex conjugate of every element:

$$ T^* = \begin{pmatrix} T_{11}^* & T_{12}^* & \cdots & T_{1n}^* \\ T_{21}^* & T_{22}^* & \cdots & T_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1}^* & T_{n2}^* & \cdots & T_{nn}^* \end{pmatrix} $$

A matrix is **real** if all its elements are real and **imaginary** if they are all imaginary:

**Real:** $T^* = T$

**Imaginary:** $T^* = -T$
Hermitian Conjugate of a Matrix: $\tilde{T}$

The **Hermitian conjugate** (or **adjoint**) of a matrix is the transposed conjugate:

$$T^\dagger = \tilde{T}^* = \begin{pmatrix} T_{11}^* & T_{21}^* & \cdots & T_{n1}^* \\ T_{12}^* & T_{22}^* & \cdots & T_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n}^* & T_{2n}^* & \cdots & T_{nn}^* \end{pmatrix}$$

$$a^\dagger = \tilde{a}^* = (a_1^*, a_2^*, \cdots, a_n^*)$$

A square matrix is **Hermitian** (or **self-adjoint**) if it is equal to its Hermitian conjugate; if Hermitian conjugate introduces a minus sign, the matrix is **skew Hermitian** (or **anti-Hermitian**).

**Hermitian:** $T^\dagger = T$

**Skew Hermitian:** $T^\dagger = -T$

The inner product of two vectors (with respect to an orthonormal basis) can be written very neatly as

$$\langle \alpha | \beta \rangle = a^\dagger b$$
Commutator

Matrix multiplication is not, in general, commutative \((ST \neq TS)\); the difference between the two orderings is called the **commutator**

\[
[S, T] = ST - TS.
\]

The transpose of a product is the product of the transposes *in reverse order*:

\[
(ST)^\top = T^\top S^\top.
\]

The same goes for Hermitian conjugate:

\[
(ST)^\dagger = T^\dagger S^\dagger.
\]
Unit Matrix $I$ & Inverse Matrix $T^{-1}$

The **unit matrix** represents a linear transformation that carries every vector into itself and consists of ones on the main diagonal and zeros everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \text{or} \quad I_{ij} = \delta_{ij}$$

The **inverse** of a matrix is defined in the obvious way:

$$T^{-1}T = TT^{-1} = I$$

A matrix has an inverse if and only if its **determinant** is nonzero; in fact,

$$T^{-1} = \frac{\tilde{C}}{\det T}$$

where $C$ is the matrix of **cofactors**. A matrix without an inverse is said to be **singular**.

The inverse of a product (assuming it exists) is the product of the inverses **in reverse order**:

$$(ST)^{-1} = T^{-1}S^{-1}$$
Unitary Matrix

A matrix is **unitary** if its inverse is equal to its Hermitian conjugate:

\[ U^\dagger = U^{-1} \]

Assuming the basis is orthonormal, it can be shown that the columns of a unitary matrix consitute an orthonormal set, and so too do its rows.