Lecture 15

Topics to be covered:

- $\delta$-function potential: bound states
- $\delta$-function potential: scattering states
- $\delta$-function potential barrier
Bound and Scattering States

Condition for bound state:

Because the phenomenon of tunneling allows the particle to “leak” through any finite potential barrier, so the only thing that matters is the potential at infinity:

\[ E < V(-\infty) \text{ and } V(+\infty) \implies \text{bound state} \]
\[ E > V(-\infty) \text{ or } V(+\infty) \implies \text{scattering state} \]

In practice, most potentials go to zero at infinity, in which case the criterion simplifies to

\[ E < 0 \implies \text{bound state} \]
\[ E > 0 \implies \text{scattering state} \]
Dirac Delta Function

The Dirac delta function, \( \delta(x) \), is defined as

\[
\delta(x) = \begin{cases} 
0, & \text{if } x \neq 0 \\
\infty, & \text{if } x = 0
\end{cases}
\]

with

\[
\int_{-\infty}^{\infty} \delta(x)dx = 1
\]

Properties:

\[
f(x)\delta(x-a) = f(a)\delta(x-a)
\]

\[
\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)dx = f(a)
\]
The Delta Function Potential

Consider

\[ V(x) = -\alpha \delta(x) \]

The Schrödinger equation

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} - \alpha \delta(x) \psi = E \psi \]

\[ E < 0 \implies \text{bound state} \]
\[ E > 0 \implies \text{scattering state} \]
Bound State

In the region $x < 0$, $V(x) = 0$ and

$$\frac{d^2 \psi}{dx^2} = -\frac{2mE}{\hbar^2} \psi$$

Let

$$\kappa = \sqrt{-\frac{2mE}{\hbar^2}}$$

then

$$\frac{d^2 \psi}{dx^2} = \kappa^2 \psi$$

The solution is

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x}$$

Boundary condition at $x \to -\infty$ requires $A = 0$

$$\psi(x) = Be^{\kappa x}, \quad (x < 0)$$
Bound State: Boundary Conditions

In the region \( x > 0 \), \( V(x) = 0 \) and the general solution is
\[
\psi(x) = Fe^{-\kappa x} + Ge^{\kappa x}
\]
However, the second term blows up at \( x \to \infty \). Thus
\[
\psi(x) = Fe^{-\kappa x}, \quad x > 0
\]

Boundary conditions

1. \( \psi \) is always continuous.
2. \( d\psi/dt \) is continuous except at points where the potential is infinite.

The first boundary condition gives
\[
F = B
\]
and
\[
\psi(x) = \begin{cases} 
Be^{\kappa x}, & (x \leq 0) \\
Be^{-\kappa x}, & (x \geq 0) 
\end{cases}
\]
Boundary Conditions

The wave function can be written as

$$\psi(x) = B e^{-\kappa |x|}$$

When $V$ is infinite at the boundary, the wave function has a kink and the derivative is not continuous.

We can integrate the Schrödinger equation from $-\epsilon$ to $+\epsilon$ and then take the limit as $\epsilon \to 0$.

$$-\frac{\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} \frac{d^2 \psi}{dx^2} dx + \int_{-\epsilon}^{\epsilon} V(x) \psi(x) dx = E \int_{-\epsilon}^{\epsilon} \psi(x) dx$$

Since $\psi(x)$ is continuous, the integral on the right hand side is zero.
Energy of the Particle

\[ \int_{-\epsilon}^{\epsilon} \frac{d^2 \psi}{dx^2} \, dx = \frac{2m}{\hbar^2} \int_{-\epsilon}^{\epsilon} V(x) \psi(x) \, dx \]

Ordinarily, the limit on the right is again zero, and \( d\psi/dx \) is continuous. For the delta function potential

\[ \frac{d\psi}{dx} \bigg|_{+\epsilon} - \frac{d\psi}{dx} \bigg|_{-\epsilon} = -\frac{2m\alpha}{\hbar^2} \psi(0) \]

Using the wave functions given on slide 6,

\[ -B\kappa - B\kappa = -2\kappa B = -\frac{2m\alpha}{\hbar^2} B \]

Thus

\[ \kappa = \frac{m\alpha}{\hbar^2} \]

and

\[ E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2} \]
Wave Function

Normalize $\psi$,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2|B|^2 \int_{0}^{\infty} e^{-2\kappa x} dx = \frac{|B|^2}{\kappa} = 1$$

$$B = \sqrt{\kappa} = \frac{\sqrt{m\alpha}}{\hbar}$$

The delta-function well, regardless of its “strength” $\alpha$, has exactly one bound state:

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$$

$$E = -\frac{m\alpha^2}{2\hbar^2}$$
**Scattering States**

Scattering states occur when $E > 0$.

For both $x < 0$ and $x > 0$, the Schrödinger equation reads

$$-rac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E\psi$$

or

$$\frac{d^2 \psi}{dx^2} + k^2 \psi = 0$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

The general solutions are

$$\psi(x) = \begin{cases} 
Ae^{ikx} + Be^{-ikx} & (x < 0) \\
Ce^{ikx} + De^{-ikx} & (x > 0) 
\end{cases}$$

In a typical scattering experiment particles are fired in from one direction — let’s say, from the left. Then $D = 0$.

$$\psi(x) = \begin{cases} 
Ae^{ikx} + Be^{-ikx} & (x < 0) \\
Ce^{ikx} & (x > 0) 
\end{cases}$$
Boundary Conditions

The continuity of $\psi(x)$ at $x = 0$ requires that

$$A + B = C \quad (1)$$

The derivative of the wave function is not continuous since the potential is infinite at $x = 0$. Direct integration of the Schrödinger equation yields

$$\left. \frac{d\psi}{dx} \right|_{+0} - \left. \frac{d\psi}{dx} \right|_{-0} = \frac{2m}{\hbar^2} \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} V(x)\psi(x)dx$$
Boundary Conditions

Since

\[
\psi(x) = \begin{cases} 
Ae^{ikx} + Be^{-ikx} & (x < 0) \\
Ce^{ikx} & (x > 0)
\end{cases}
\]

\[
\frac{d\psi}{dx} \bigg|_{+0} = ikC \\
\frac{d\psi}{dx} \bigg|_{-0} = ik(A - B)
\]

The right hand side

\[
\frac{2m}{\hbar^2} \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} V(x)\psi(x)dx = -\frac{2m\alpha}{\hbar^2} \psi(0) = -\frac{2m\alpha}{\hbar^2} C
\]

Thus

\[
ikC - ikA + ikB = -\frac{2m\alpha}{\hbar^2} C
\]

\[
A - B = \left(1 - \frac{2im\alpha}{\hbar^2 k'}\right) C \quad (2)
\]
Boundary Conditions

Eqs. (1) and (2) can be used to solve $B$ and $C$ in terms of $A$. Adding and subtracting the two equations, we get

$$A = \left(1 - \frac{im\alpha}{\hbar^2 k}\right) C$$

$$B = \frac{im\alpha}{\hbar^2 k} C$$

Therefore

$$\frac{C}{A} = \frac{1}{1 - im\alpha/\hbar^2 k}$$

$$\frac{B}{A} = \frac{BC}{CA} = \frac{im\alpha/\hbar^2 k}{1 - im\alpha/\hbar^2 k}$$

The reflection coefficient and the transmission coefficient are then given by

$$R = \left|\frac{B}{A}\right|^2 = \frac{1}{1 + (\hbar^2 k/m\alpha)^2} = \frac{1}{1 + (2\hbar^2 E/m\alpha^2)}$$

$$T = \left|\frac{C}{A}\right|^2 = \frac{1}{1 + (m\alpha/\hbar^2 k)^2} = \frac{1}{1 + (m\alpha^2/2\hbar^2 E)}$$
**Delta-function Barrier**

Can be obtained from the delta-function potential well by changing the sign of $\alpha$.

Since the reflection and transmission coefficients depend on $\alpha^2$, they remain unchanged.

The particle is just as likely to pass through the barrier as to cross over the well!