Wave Function

de Broglie wave:

\[ E = h\nu = \hbar\omega \]

\[ \vec{p} = \frac{\hbar}{\lambda} = \hbar\vec{k} \]

Free particle

\[
\begin{align*}
E &= \text{constant} \\
\vec{p} &= \text{constant vector}
\end{align*}
\]

\[ \Downarrow \]

Constant frequency and wave length

\[ \Downarrow \]

Plane Wave!
Free Particle & Plane Wave

A plane wave, with frequency $\nu$ and wavelength $\lambda$ and travelling along the $x$-axis can be described by

$$\psi(x, t) = A \cos \left[ 2\pi \left( \frac{x}{\lambda} - \nu t \right) \right]$$
or

$$\psi(x, t) = A \cos (kx - \omega t)$$

For a wave travelling along $\hat{n}$,

$$\psi(\vec{r}, t) = A \cos \left[ 2\pi \left( \frac{\vec{r} \cdot \hat{n}}{\lambda} - \nu t \right) \right]$$
or

$$\psi(\vec{r}, t) = A \cos \left( \vec{k} \cdot \vec{r} - \omega t \right)$$

In complex form,

$$\psi(\vec{r}, t) = Ae^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Only the complex form can be used to describe free particle in quantum mechanics.
Interpretation of Wave Function

In general, the momentum and energy of a particle are not constants. The particle cannot be described by a plane wave. A more general function, $\Phi(x, y, z, t)$ or $\Phi(\mathbf{r}, t)$ is required to describe the particle wave. $\Phi$, a complex function, is called the wave function.

*How to understand the relationship between the wave function and the particle described by the wave?*
Interpretation of Wave Function

Electron Diffraction:

High intensity beam $\implies$ fast image formation
Low intensity beam $\implies$ slow image formation
Very low intensity beam $\implies$ From scattered dots to image

_The image (interference pattern) is the same!_

The wave property of electron is the statistical result of many electrons in the same experiment.

Or equivalently

The wave property of electron is the statistical result of many repetitions of the same experiment with the electron.
Born’s Statistical Interpretation

The intensity of the wave function (the absolute square of the wave function) at a given point \( \vec{r} \) and time \( t \) is proportional to the probability of finding the particle at the same point and time.

Let \( dW(x, y, z, t) \) be the probability of finding the particle in a volume element between \( x \) and \( x + dx \), \( y \) and \( y + dy \), and \( z \) and \( z + dz \) at time \( t \), then \( dW(x, y, z, t) \) must be proportional to the volume element \( d\tau = dx dy dz \) and the absolute square of the wave function, according to the statistical interpretation of the wave function, i.e.,

\[
dW(x, y, z, t) = C|\Phi(x, y, z, t)|^2 d\tau
\]

where \( |\Phi|^2 = \Phi^*\Phi \), and \( \Phi^* \) is the complex conjugate of \( \Phi \).

This is a probability wave!
# Born’s Statistical Interpretation

**Electron Diffraction**

<table>
<thead>
<tr>
<th>Diffraction maximum</th>
<th>Diffraction minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>High intensity</td>
<td></td>
</tr>
<tr>
<td>Large probability of finding electrons</td>
<td></td>
</tr>
<tr>
<td>more electrons</td>
<td></td>
</tr>
<tr>
<td>Low (or zero) intensity</td>
<td></td>
</tr>
<tr>
<td>Small (or zero) probability of finding electrons</td>
<td></td>
</tr>
<tr>
<td>less or no electrons</td>
<td></td>
</tr>
</tbody>
</table>

![Electron Diffraction Pattern](image)
Born's Statistical Interpretation

Classical physics:

Variables: \( x \) and \( p \).

Mechanical properties: \( E(x, p) \).

Quantum mechanics:

\( \psi(x, y, z, t) \) describes the quantum state of the particle.

Due to particle-wave duality, \( x \) and \( p \) cannot be determined at the same time when the particle is in a given quantum state, its mechanical quantities can have many possible values, each with a certain probability, given by the wave function.
Born's Statistical Interpretation

Probability Density:

\[ w(x, y, z, t) = \frac{dW(x, y, z, t)}{d\tau} = C|\Phi(x, y, z, t)|^2 \]

The probability of finding the particle in the entire space,

\[ \int_{\infty} dW(x, y, z, t) = C \int_{\infty} |\Phi(x, y, z, t)|^2 d\tau \]

This must be equal to \textbf{ONE} since the particle got to be somewhere.

\[ C \int_{\infty} |\Phi(x, y, z, t)|^2 d\tau = 1 \]

\[ \Rightarrow C' = \frac{1}{\int_{\infty} |\Phi(x, y, z, t)|^2 d\tau} \]
Normalization of Wave Function

Probability depends on the relative intensity of the wave function. The absolute intensity is insignificant (different from light and sound waves!).

If $\Phi$ is the wave function of a particle, $C\Phi$ can be used to describe the same particle as long as $C$ is a constant. We can choose $C$ such that

$$\Psi(x, y, z, t) = \sqrt{C}\Phi(x, y, z, t)$$

and

$$\int_{-\infty}^{\infty} |\Psi(x, y, z, t)|^2 d\tau = 1$$

$$dW(x, y, z, t) = |\Psi(x, y, z, t)|^2 d\tau$$

$$\omega(x, y, z, t) = |\Psi(x, y, z, t)|^2$$

The wave function $\Psi(x, y, z, t)$ satisfying the above equations is said to be **normalized**. $C$ is called a normalization constant.

A normalized wave function is still not unique. It can be multiplied by $e^{i\delta}$.

Plane wave cannot be normalized in this way.