PC5202 Advanced Statistical Mechanics

Assignment 6 (due Thursday 9 Apr 20)

1.
(a) The scaling of the singular part of the free energy per spin near a critical point $t=(T-T_c)/T_c=0$ and $h=0$ is

$$f(t,h)=b^{-d}f(b^Y t, b^X h).$$

Similar equation holds for the correlation length

$$\xi(t,h)=b\xi(b^Y t, b^X h).$$

Using the definition that $\xi(t,0)\propto t^{-\nu}$, show that $Y=1/\nu$.
(b) Show that the total susceptibility for an Ising model can be expressed as an (*d*-dimensional) integral of the correlation function in the continuum limit:

$$\chi=\frac{\partial M}{\partial h}\approx \beta N\int d^d\mathbf{r}\,G(r),\quad \beta=\frac{1}{k_BT},$$

$$G(r)=\langle\sigma_i\sigma_{i+r}\rangle-\langle\sigma_i\rangle\langle\sigma_{i+r}\rangle.$$

(c) The correlation function *G* satisfies the scaling relation

$$G(r,\xi)=b^{(-d+2-\eta)}G(r/b,\xi/b).$$

Using this scaling relation and result of (b), prove the Fisher scaling law:

$$\gamma=\nu(2-\eta),$$

where the susceptibility exponent γ is defined by $\chi\propto t^{-\gamma}$.

2. Consider the Langevin equation with a constant driven force

$$m\frac{dv}{dt}=-m\gamma v+f+R(t),$$

$$\langle R(t)\rangle=0,$$

$$\langle R(t)R(t')\rangle=C\delta(t-t').$$

(a) Determine the average velocity $\bar{v}(t)=\langle v(t)\rangle$.
(b) Determine the mean square deviation $\left\langle\left(v(t)-\bar{v}(t)\right)^2\right\rangle$.
(c) Determine the four-time correlation function $\langle\delta v(t_1)\delta v(t_2)\delta v(t_3)\delta v(t_4)\rangle$, where $\delta v(t)=v(t)-\bar{v}(t)$.
(d) Find the associated Fokker-Planck equation for the probability distribution $P(v,t)$ of the velocity.

3. The Kubo canonical correlation is defined by

$$\langle a;b\rangle=\frac{1}{\beta}\int_0^\beta d\lambda\,\mathrm{Tr}\left(\rho e^{\lambda H}a\,e^{-\lambda H}b\right).$$

Show that $\beta\langle \dot{a};b\rangle = \frac{1}{i\hbar}\langle [a,b]\rangle$,

where $\beta=1/(k_B T)$, a and b are arbitrary quantum operators, $[a, b] = ab - ba$ is the commutator , and $\dot{a} \equiv \frac{1}{i\hbar}[a,H]$, H is the Hamilton operator of the system, the angular brackets denote quantum average with respect to canonical ensemble density matrix $\rho = \exp(-\beta H)/Z$. [Hint: use the Kubo identity, see e.g., R. Kubo, M. Toda, and N. Hashitsume, Statistical Physics II, Chap.4]

Tutorial 6

4. (Plischke & Bergersen, Ch.6.1) Renormalization group transformation for the 1D Ising model. We will show the steps of decimation renormalization group transform for the 1D model, which can be done exactly.

5. Consider the Langevin equation in one dimension

$$m\frac{d^2x}{dt^2} = -kx - m\gamma\frac{dx}{dt} + R(t),$$
$$\langle R(t)\rangle = 0,$$
$$\langle R(t)R(t')\rangle = C\delta(t'-t).$$

This is a harmonic oscillator with mass m and spring constant k, subjected to damping and a random force (white noise).

(a) Let the Fourier transform of the coordinate x be

$$\tilde{x}[\omega] = \int_{-\infty}^{+\infty} x(t)e^{i\omega t}dt\ .$$

Derive the algebraic equation that the Fourier component $\tilde{x}[\omega]$ must satisfy; solve the equation in terms of the Fourier transform of the random noise.

(b) Find the expression, $\tilde{F}[\omega]$, in terms of the model parameters (m, k, γ, C) and frequency ω, which is the Fourier transform of the correlation function, $F(t) = \langle x(t)x(0)\rangle$, [Hint: you may use the Wiener-Khintchine theorem].

(c) Use the inverse Fourier transform to find $F(t)$.

6. Consider the generalized Langevin equation

$$m\frac{dv(t)}{dt} = -\frac{dV(x)}{dx} - m\int_{-\infty}^{t} \gamma(t-t')v(t')dt + R(t),$$

$$\frac{dx(t)}{dt} = v(t).$$

(a) Compute the time change of kinetic energy plus potential energy,

$$\frac{dE(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + V\right) .$$

(b) Show that the work done by the noise and dissipative force (the friction force) are equal for sufficiently long time.