PC5202 Advanced Statistical Mechanics

Assignment 4 (due Thursday 19 Mar 2020)

1. For an Ising paramagnetic model with the energy

$$H(\sigma) = -h\sum_{i=1}^{N}\sigma_i, \quad \sigma_i = \pm 1,$$

show that the entropy can be expressed as

$$S = Nk_B\left[-\frac{1+m}{2}\ln\frac{1+m}{2} - \frac{1-m}{2}\ln\frac{1-m}{2}\right]$$

where $m = \langle \sigma_i \rangle$ is the magnetization per spin.

2. (Plischke & Bergersen, Chap 3.8) Consider a Landau theory with an assumption of the Gibbs free energy as

$$G(T,M) = G(T,0) + a(T - T^*)M^2 - \frac{1}{3}cM^3 + \frac{1}{4}dM^4$$

where *a*, *c*, and *d* are some positive constants.

(a) Show that the order parameter *M* is discontinuous at some transition point T_c. Determine T_c. Draw a qualitatively correct curve *M* vs. temperature *T*. Pay attention to the stability of the solutions. The stable solution should be a global minimum of *G* with respect to *M*.

(b) Find the latent heat *Q* of the first-order phase transition across T_c.

3. (From question 4.4 of J M Yeomans, page 64) The mean-field equations for the three-state Potts model,

$$H(\sigma) = -\frac{3}{2}J\sum_{\langle i,j\rangle}\delta_{\sigma_i\sigma_j}, \quad \sigma_i = 1,2,3,$$

can be derived as follows using the result obtained for vector spin model.

(a) Show that the Potts model is equivalent to $H = -J\sum_{\langle i,j\rangle}\vec{s}_i \cdot \vec{s}_j + \text{const}$ where

$\vec{s}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$ are two-dimensional unit vectors.

(b) Putting $\vec{B}_0 = \begin{pmatrix} B_0 \\ 0 \end{pmatrix}$ show that the mean-field equations become

$$\vec{B}_0 = zJ\langle \vec{s}\rangle_0,$$

$$\frac{B_0}{zJ} = \frac{e^{3\beta B_0/2} - 1}{e^{3\beta B_0/2} + 2},$$

$$F_{MF} = -Nk_BT\ln\left(e^{\beta B_0} + 2e^{-\beta B_0/2}\right) + \frac{NB_0^2}{2zJ},$$

where the subscript 0 denotes an average in the ensemble defined by $H_0 = -\vec{B}_0 \cdot \sum_i \vec{s}_i$, N is the number of spins, and z is the coordination number of each spin.

(c) Expand F_{MF} in $\langle s \rangle_0$ and show that it contains a cubic term. By sketching the mean-field free energy for suitable values of the coefficients in the expansion show that the transition is first-order.

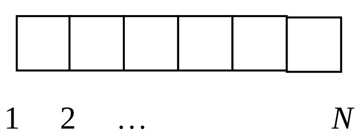
(d) Verify that the transition is at $k_B T_C = 3zJ/(8\ln 2)$ and that the jump in the magnetization is $1/2$.

Tutorial 4

4. Consider an Ising model defined on a ladder (i.e., a $2\times N$ lattice) without the magnetic field:

$$E = -J\sum_{\langle ij \rangle} \sigma_i \sigma_j ,$$

where the site i runs over the 2 by N lattice; the summation is over the nearest neighbor only, and each site has three neighbors. (a) Give the 4×4 transfer matrix P such that the partition function $Z = \mathrm{Tr}(P^N)$. (b) Diagonalize P to find an analytic expression for the maximum eigenvalue λ of P. (c) Show that the free energy $F = -Nk_B T \ln \lambda$ in the thermodynamic limit.



5. Consider the Ising model defined on a d-dimensional hyper-cubic lattice with

$$E = -J\sum_{\langle ij \rangle} \sigma_i \sigma_j - h\sum_i \sigma_i .$$

(a) Show that the magnetic susceptibility in the limit of $h\to 0$ is given by

$$\chi \equiv \left.\frac{\partial \langle M \rangle}{\partial h}\right|_{h\to 0^+} = \frac{1}{k_B T}\left[\langle M^2 \rangle - \langle M \rangle^2\right], \quad M = \sum_{i=1}^{N} \sigma_i .$$

(b) Compute the first two nonzero terms in a high-temperature expansion of the susceptibility in the variable $x = \tanh(\beta J)$, $\beta = 1/(k_B T)$.