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[14]A. H. Compton, *Atomic Quest* (Oxford University, New York, 1956), p. 192.

Fundamental resonant frequency of a loudspeaker

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The fundamental resonant frequency of a loudspeaker marks its useful low-frequency limit. Its values may be determined by plotting the electrical impedance of the loudspeaker against frequency. This was done for a loudspeaker under various loading conditons. The graphs obtained yield useful information about the loudspeaker's mass, compliance, and other parameters.

Loudspeaker theory is usually one of the more interesting topics in an undergraduate acoustics course, because of the students' interest in high fidelity and sound reproduction. Many students will themselves have had practical experience in designing and constructing their own loudspeaker enclosures, and the use of loudspeaker theory to highlight general acoustics topics such as acoustical impedance greatly enhances the attractiveness of these topics for the students.

Laboratory experiments on loudspeakers in conjunction with such topics would be of interest for the same reasons. However, the usual means of measuring loudspeaker parameters such as frequency response has required the use of expensive and often unobtainable anechoic chambers. Impulse techniques utilizing fast Fourier transforms have made loudspeaker testing without anechoic chambers possible. On the other hand, these newer techniques also require expensive instrumentation as well as the application of mathematical techniques that may obscure the acoustical principles for the student.

There is one useful loudspeaker parameter that can be easily and inexpensively measured by undergraduate students. The fundamental resonant frequency f_r of a loudspeaker can be determined using simple instrumentation, yet it can give important information on loudspeaker behavior. In fact, f_r marks the effective low-frequency performance limit of a loudspeaker system, and hence is often quoted in manufacturer's loudspeaker specifications.

IMPEDANCE OF A DYNAMIC LOUDSPEAKER

A dynamic or direct-radiator loudspeaker consists of a stiff and light cone suspended from a rigid metal frame. The cone is driven by a voice coil of length l attached to the end of the cone. The current i passing through the coil is perpendicular to the radial magnetic field B of a permanent magnet. The resultant driving force F on the cone is equal to Bli.

The cone and voice coil have a combined mass M and the cone suspension has a compliance C that together form a mechanical system. The system can be represented by an analogous electrical circuit in which mechanical quantities are represented by electrical quantities; force by emf, velocity by current, mass by inductance, compliance by capacitance, and mechanical resistance by electrical resistance. The resultant velocity v due to a force F is given by

$$v = F/Z_M,$$

where Z_M is defined as the mechanical impedance of the system. Z_M may be calculated from M and C from the analog circuit (Fig. 1).

FUNDAMENTAL RESONANT FREQUENCY

If the driving force F is oscillating, the magnitude of Z_M is at a minimum when the frequency f is equal to the fundamental resonant frequency f_r of the loudspeaker where

$$f_r = 1/2\pi(MC)^{1/2}.$$

The resultant velocity is at a maximum when this condition holds.

The above assumes that the loudspeaker is oscillating in

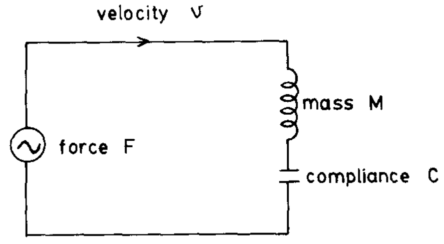

Fig. 1. Electrical analog circuit of a loudspeaker in vacuum.

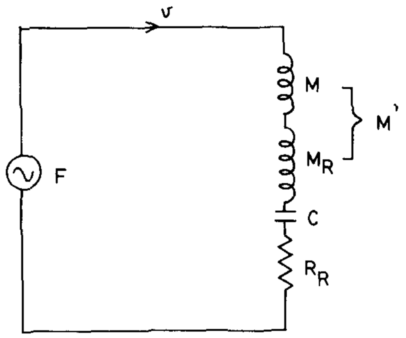

Fig. 2. Electrical analog circuit of a loudspeaker in air.

a vacuum, and that there are no dissipative effects. Under practical conditions, the loudspeaker will be in a fluid medium such as air. The values of Z_M and f_r will be different in air for the following reasons:

(a) The mass M is effectively increased to a value M' due to the air loading M_R on the cone.

(b) The radiation of acoustical power from the loudspeaker through the medium is equivalent to the dissipation of power through a "radiation resistance" R_R in series with M and C (Fig. 2).

The square of f_r will decrease inversely with the increase in effective mass from M to M'.

When the loudspeaker is placed under different mounting conditions, e.g., in a box, the loading conditions will be different resulting in changes in Z_M and f_r.

BACK EMF AND MOTIONAL IMPEDANCE

We turn now to the electrical circuit of the loudspeaker coil. The coil is driven by an emf e giving rise to the current i and hence the force F. The resultant motion of the coil in the field B generates a back emf e_B opposite to e, given by

$$e_B = Blv.$$

Hence

$$e_B = B^2l^2i/Z_M.$$

The current i is given by

$$i = (e - e_B)/Z_E,$$

where Z_E is the total electrical impedance in the coil circuit. Substituting for e_B, we can show that

$$i = e/(Z_E + Z_{mot}),$$

where

$$Z_{mot} = B^2l^2/Z_M.$$

Z_{mot} is the additional effective electrical impedance in the electrical circuit due to the motion of the mechanical system of the loudspeaker, and hence is known as the motional impedance.[1,2]

DETERMINATION OF f_r

At the frequencies of interest, near the fundamental resonant frequency of the loudspeaker, Z_E does not vary greatly in comparison to Z_M. Hence by determining the variation of e/i with frequency, the frequency dependance of Z_{mot} and hence of Z_M can be obtained. In particular, a maximum in e/i or Z_{mot} corresponds to a minimum in Z_M, and hence gives the value of f_r.

As stated earlier, the importance of f_r is due to the fact that it marks the effective low-frequency response limit of a loudspeaker. This method of determining f_r is purely electrical and very simply carried out.

EXPERIMENTAL PROCEDURE AND RESULTS

The measurement of the ac voltage–current ratio for the loudspeaker coil requires nothing more than an audio oscillator, an ac voltmeter, and an ac ammeter. It is instructive to plot e/i against oscillator frequency and obtain f_r for a loudspeaker under the following conditions: (a) in a vacuum; (b) free-standing in air; (c) free-standing in air with small masses symmetrically attached to the cone; (d) mounted in a totally enclosed box; (e) mounted in a totally enclosed box filled with acoustically absorbent material.

It should be emphasized that these impedance–frequen-

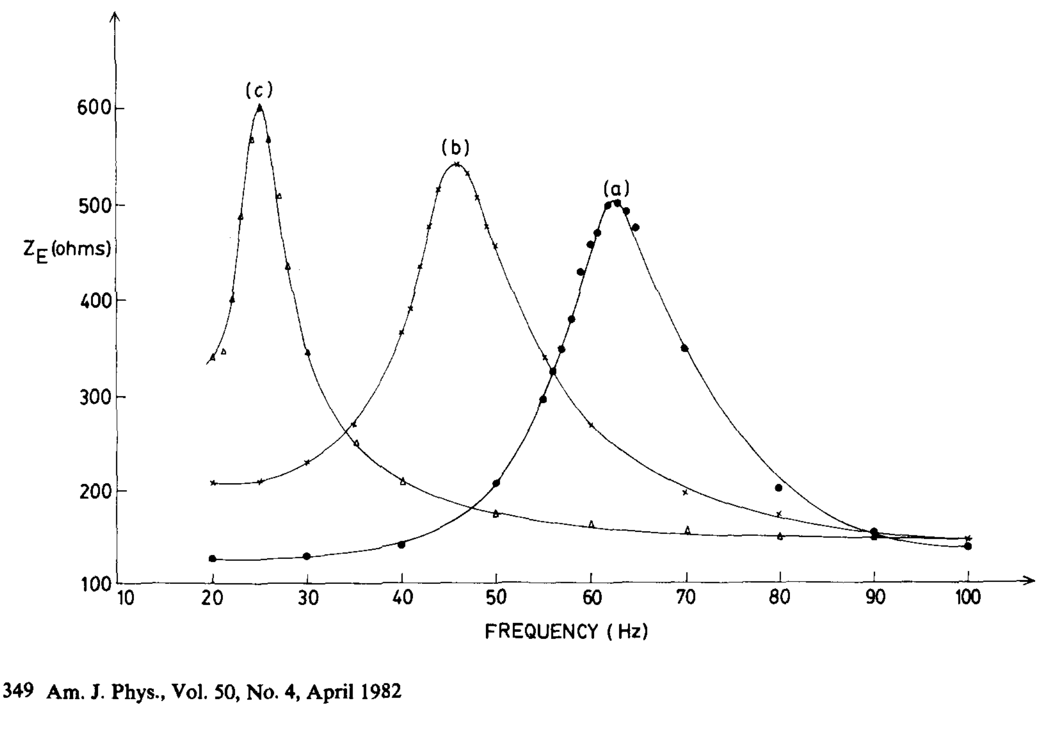

Fig. 3. Graphs of electrical impedance versus frequency for an 8-in. loudspeaker under following conditions: (a) in vacuum; (b) free standing in air; (c) as (b) with masses attached to the cone.

cy plots are different from the output power–frequency response plots of the loudspeaker. Even a high-fidelity loudspeaker with a flat frequency response will display a peak at f_r in its impedance–frequency plot. As the ratio e/i is being measured, the driving conditions need not be purely constant current or constant voltage, provided the loudspeaker is not being overdriven to distortion. Acoustic output power is dependent on the driving condition, but the motional impedance is dependent only on the driving frequency.[3] The loudspeaker used in all the experiments was a nominally 8-in. dia. paper-cone dynamic loudspeaker.

Experiments (a), (b), and (c) may be discussed together (Fig. 3). In (b) the effective cone mass is M' (including M_R, the air loading mass). In (c), two small and similar pieces of plasticine (say 1 cm in diameter) or solder, taped to the cone opposite each other will suffice. The combined mass of the plasticine used was 10 gm. This extra mass gives rise to a decrease in f_r from its value in (b). M' can then be determined from the ratio of the values of f_r in (b) and (c), and the known extra mass. The loudspeaker compliance C can then be determined from the formula for f_r. This is the conventional method of determining M' and C.

In (a), the effective mass is just M, as there is no air loading. Hence f_r will increase from its value in (b). Since M' has now been determined, M and hence M_R can be determined from the ratio of f_r in (a) and (b). The value of M_R can be compared with its theoretical value of $(8/3)\,a^3\rho_0$, where a is the cone radius and ρ_0 is the density of air. For (a), all that is necessary for the vacuum is to make the air loading negligible. The loudspeaker was placed in an air-tight wooden box and a rotary vacuum pump used to evacuate the box to a pressure less than 3 mm Hg.

The experiment (d) requires a small rigid wooden box similar in size to a small loudspeaker "bookshelf" enclosure. In the experiment, a box measuring 59×34.5×31 cm was used. A circular hole of diameter equal to that of the cone was cut in one face of the box. The loudspeaker was mounted in the box by bolting the metal frame securely to the face so that the cone faced outward through the circular hole, in the usual manner for mounting loudspeakers. The compliance of the air in the box C_B is then acoustically in parallel with that of the cone suspension, C. The combined compliance C' is less than that of C alone. In the equivalent analog circuit, C_B is in series with C. As a result, f_r is increased and the useful low-frequency range of the system is degraded (Fig. 4).

C' and hence C_B can be determined from the increase in f_r by using the ratio of f_r for (b) and (d) and the previously determined value of C. The value for C_B may be compared with the expected value given by

$$C_B = V_B/\gamma P_0 S^2,$$

where V_B is the box volume, P_0 the static air pressure, S the cone area, and γ the ratio of specific heats for air.

Finally, the practice of most loudspeaker system manufacturers of filling the box with acoustically absorbent material is investigated in (e) (Fig. 4). It is commonly assumed that this is mainly to absorb unwanted reflections in the box. The main purpose is actually to effect a lowering of f_r by increasing C_B. The acoustical material slows down the air velocity in the box and makes the acoustical expansions and rarefactions isothermal rather than adiabatic, hence altering the value of γ and hence of C_B.[4] Cotton wool was used as the absorbent material in the experiment, but many other similar materials will do as well.

The audio frequency in all the above experiments was obtained by calibrating the dial of the audio oscillator against the main's 50-Hz frequency. The calibration

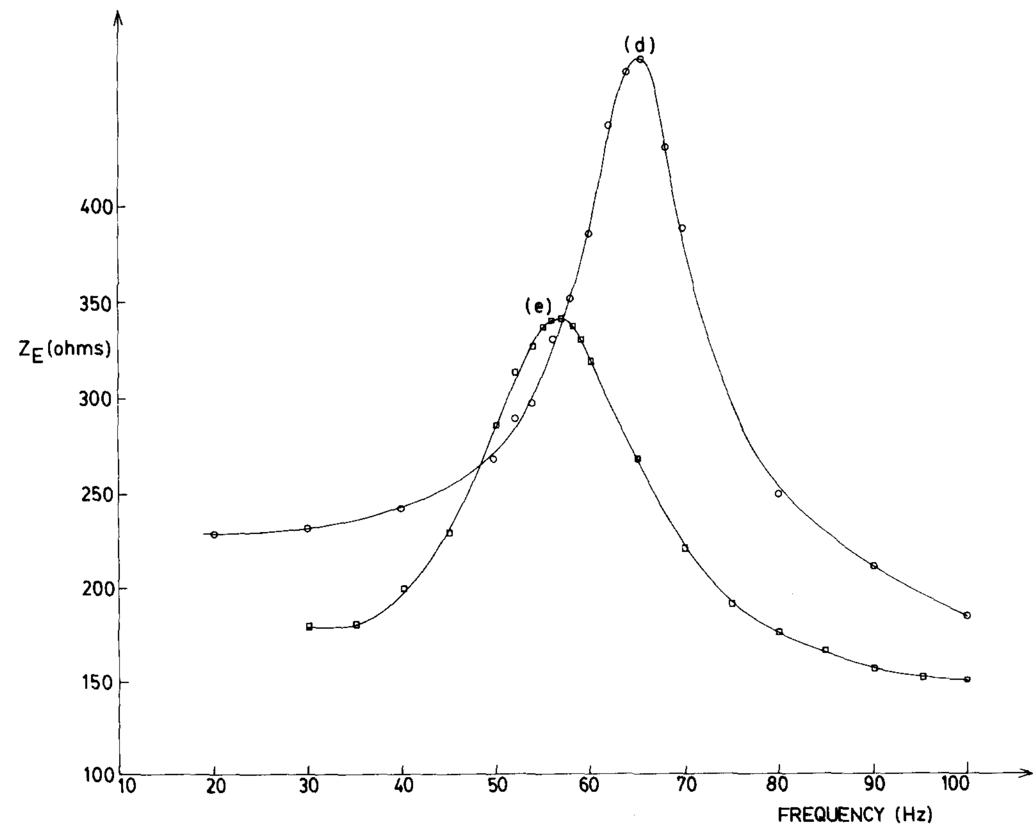

Fig. 4. (d) Same loudspeaker mounted in totally enclosed box 59×34.5×31 cm. (e) as (d), box filled with absorbent material.

Table I. Experimental values.

	f_r (Hz)	Total mass ($\times 10^{-3}$ kg)	Total compliance ($\times 10^{-3}$ m/N)
(a) vacuum	62.5	2.40	2.69
(b) free mounted	46	4.44	2.69
(c) free mounted + 10 g	25.5	14.44	2.69
(d) in box	65	4.44	1.35
(e) in box with material	57	4.44	1.76

showed the dial to be accurate to 5%.

Table I gives the values of f_r, total mass, and total compliance as obtained for each of the experiments (a) to (e). The free-mounted mass was first determined by experiments (b) and (c), and the free-mounted compliance then determined. The mass in vacuum was then determined from (a). The changes in compliance due to the box and the acoustic material were then determined from (d) and (e).

Apart from f_r, the graphs of electrical impedance against frequency may yield information about the mechanical impedance of the loudspeaker. At these relatively low frequencies, the electrical impedance of the coil may be taken to be its dc value. Thus Z_{mot} and hence Z_M may be determined and compared with the values calculated from M and C, provided B and l are also known.

The Q of the mechanical system may also be deduced from the shape of the graphs. As in electrical systems, a sharp peak indicates a high Q and hence low mechanical resistance, while a shallow peak indicates the reverse. For example, the graph for (e) shows a lower Q than that for (d), indicating that the acoustic material in the box increased the mechanical and acoustical resistance in the system.

CONCLUSION

The determination of f_r by plotting the variation of motional impedance with frequency of a loudspeaker can thus yield useful information on the low-frequency performance of a loudspeaker. As it is often the low-frequency performance that is of prime interest for a loudspeaker system, these experiments are of interest to loudspeaker enthusiasts among the students. It is also possible to extend the experiments for other loudspeaker configurations. In particular, the characteristic double peak for a properly tuned bass reflex system can also be demonstrated. It is hoped that others will find these experiments of interest in acoustics courses.

[1]L. L. Beranek, *Acoustics* (McGraw-Hill, New York, 1954), pp. 83–88 and 183–190.

[2]L. E. Kinsler and A. R. Frey, *Fundamentals of Acoustics* (Wiley, New York, 1962), pp. 247–259.

[3]Reference 2, pp. 256–257.

[4]Reference 2, pp. 270–272.

Two theorems in classical mechanics

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Two simple results are proved for a set of physical quantities that are the classical analog of the "complete set of commuting observables."

I. INTRODUCTION

The border line of quantum and classical mechanics has always been a source of insights into the conceptual structure of both the theories. The most striking feature is the so-called analogy between the classical and quantum equations, between Poisson brackets and commutators, and so on.

In 1917, Einstein[1] gave a very compelling geometric argument for the discreteness of energy (and other physical variables) in the course of his attempt to generalize the Bohr–Sommerfeld quantum conditions. This argument was later developed by Synge[2] for the relativistic case. The essential feature of this argument is a theorem, a restricted form of which was stated in "notes added to the proof" by Einstein, but not proved by him, and has not been treated by Synge. I think, for pedagogical reasons one should clearly state and prove this theorem, especially so as the proof happens to be very simple.[3] This is done in Sec. II.

Another aspect of the classical–quantum analogy is the algebraic similarity of the Poisson bracket, and the commutator. Now, an important concept in quantum mechanics is that of a complete set of commuting observables introduced by Dirac. A theorem in quantum mechanics says