NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA1506 Mathematics II

Nov/Dec 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in this paper. Marks for each question are indicated at the beginning of the question.

3. Candidates may use non-graphing, non-programmable calculators. However, they should lay out systematically the various steps in the calculations.

4. One A4 handwritten double-sided helpsheet is allowed.

Question 1 [15 marks]

(a) Consider the differential equation

$$\frac{dy}{dt} = (y^2 + 2y)(1 + y)^2.$$

(i) Find and analyse all equilibrium points.

(ii) Give a sketch showing the qualitative behaviour of all solutions to the differential equation.

(b) Let $y(t)$ be the solution to the initial value problem

$$\frac{dy}{dt} = (y^2 - 4)ln(t^2 + 1), \qquad y(3) = 1.$$

Use the no crossing principle to prove that $-2 < y(t) < 2$ for all t.

Question 2 [15 marks]

(a) (i) Given that $y(t) = 2t(t-1)e^t$ is a solution to the differential equation

$$\frac{d^2y}{dt^2} + Ay = 8te^t.$$

Find the value of A.

(ii) With the value of A found in (i), find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + Ay = t + 8te^t.$$

(b) The general solution to the differential equation

$$\frac{d^2y}{dx^2} - y = -1$$

is given by

$$y(x) = Ce^x + De^{-x} + 1.$$

If $y(0) = 1$ and $\frac{dy}{dx}(0) > 0$, prove that $C > 0$ and $\lim_{x\to\infty} y(x) = \infty$.

Question 3 [15 marks]

(a) Suppose that the differential equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0$$

models a LRC circuit, where L, R and C are positive constants. Let $I(t)$ be a solution to the differential equation. Find $\lim_{t\to\infty} I(t)$. Justify your answer.

(b) Let $\delta(t)$ be the Dirac delta function and let $x(t)$ be the solution to the initial value problem

$$\frac{d^2x}{dt^2} + 9x = -3\delta(t - \frac{\pi}{2})$$

with $x(0) = 1$ and $\frac{dx}{dt}(0) = 0$. Prove that $x(t) = 0$ for all $t > \pi/2$.

Question 4 [15 marks]
A cantilevered beam of length L, made up of extremely strong and light material, is horizontal at the end where it is attached to a wall and carries a load of 1 Newton at $x = L$ and another load of 2 Newtons at $x = \frac{L}{2}$, where x is the distance from the fixed end of the beam. The deflection $y(x)$ of the beam is given by

$$EI\frac{d^4y}{dx^4} = -2\delta(x - \frac{L}{2}) - \delta(x - L),$$

where δ is the Dirac delta function and E and I are constants. Given that

$$y(0) = \frac{dy}{dx}(0) = 0$$

and

$$\frac{d^2y}{dx^2}(L) = \frac{d^3y}{dx^3}(L) = 0.$$

Use Laplace transform to calculate the deflection at $x = \frac{L}{3}$, leaving your answer in terms of E, I and L.

Question 5 [20 marks]

(a) Consider the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If $a + d < 0$ and $ad - bc > 0$, prove that all solutions to the system must approach zero as t approaches infinity.

(b) Consider the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(i) Determine the eigenvalues in terms of a.

(ii) Find the range of values of a such that the equilibrium point is a nodal sink.

(c) Consider the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(i) Classify the equilibrium point (points).

(ii) Sketch the phase plane for the system. Your sketch should have enough solution curves to show what happens in every part of the phase plane. Use arrows to indicate the direction of motion for all solutions

Question 6 [20 marks]

(a) Two tanks A and B, each holding 50 litres of liquid, are interconnected by pipes with liquid flowing from tank A into tank B at a rate of 4 litres per minute and from tank B into A at 1 litre per minute. The liquid inside each tank is well stirred. Pure water flows into tank A at a rate of 3 litres per minute and the solution flows out (and discarded) from tank B at a rate of 3 litres per minute. If, initially, tank A contains $25kg$ of salt and tank B contains no salt (only pure water), determine the mass of salt in each tank after t minutes.

(b) A company has two interacting branches, B_1 and B_2. Branch B_1 consumes \$0.5 of its own output and \$0.2 of B_2 output for every \$1 it produces. Branch B_2 consumes \$0.6 of B_1 output and \$0.4 of its own output for every \$1 it produces.

The company has a monthly external demand of \$50,000 for B_1 product and \$40,000 for B_2 product. Find a production schedule to meet the external demand.

(c) The population $y(t)$ of a certain species of fish in a lake in which fishing is allowed can be modelled by the following differential equation

$$\frac{dy}{dt} = y(M - y) - hy,$$

where h, M are positive constants. If $0 < h < M$, find the limiting population (in terms of h and M) of the fish in the lake. Justify your answer.

END OF PAPER