NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2011-2012

MA1506 Mathematics II

Nov/Dec 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **SIX (6)** printed pages.

2. Answer **ALL** questions in this paper. Marks for each question are indicated at the beginning of the question. The maximum score for this examination is **80 marks**.

3. Candidates may use non-graphing, non-programmable calculators. However, they should lay out systematically the various steps in the calculations.

4. Two A4 handwritten double-sided helpsheets are allowed.

5. This is a **CLOSED BOOK** examination.

Answer all the questions.

Marks for each question are indicated at the beginning of the question.

Question 1 [20 marks]

(a) A tank has 30 litres of salt solution and the concentration of salt in the solution is 1 kg per litre. Starting at time $t = 0$, a salt solution containing 0.5 kg of salt per litre is being poured into the tank at a constant rate of 2 litres per minute. Meanwhile, the well-mixed solution is constantly being pumped out of the tank at a constant rate of 3 litres per minute.

(i) Find the volume of the salt solution in the tank when $t = 20$ minutes.

(ii) Find the amount of salt in the tank when $t = 20$ minutes.

(b) The motion of a forced undamped oscillator system is governed by the differential equation

$$\ddot{x} + kx = F\cos(\alpha t), \qquad x(0) = \dot{x}(0) = 0.$$

Given that the solution to the differential equation is

$$x(t) = \frac{F}{k - \alpha^2}[\cos(\alpha t) - \cos(\sqrt{k}\,t)].$$

Write down the MATLAB code to plot the solution curve $x(t)$ from $t = 0$ to eight times the period of the natural frequency when $k = 9, F = 80, \alpha = 5$. Include the name MA1506 as title of your graph.

(c) Find the range of values of k such that all non-zero solutions to the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + ky = 0$$

will either tend to infinity or minus infinity as x tends to infinity.

(d) Let $u(x,t) = x^4 - 6x^2t^2 + t^4$ for all real numbers x and t. Find the value of

$$u_{xx}(x,t) + u_{tt}(x,t).$$

Question 2 [20 marks]

(a) Consider the differential equation

$$\frac{d^2y}{dx^2} - y = -1.$$

(i) Find the general solution to the differential equation.

(ii) Let $y(x)$ be a solution to the differential equation such that $y(0) = 1$ and $y'(0) < 0$. Prove that $\lim_{x\to\infty} y(x) = -\infty$.

(b) An unit mass is attached to the free end of a light spring with spring constant 4. It is initially at rest in its equilibrium position. Starting at time $t = 0$ second, an external force $F(t) = \cos(2t)$ is applied to the mass, but at time $t = 2\pi$ seconds, this external force is removed and the mass is allowed to continue its motion unimpeded. Describe its motion, assuming there is no damping in the motion.

(c) A uniform cantilever beam of length L is fixed at $x = 0$ and free at its other end $x = L$. The deflection $y(x)$ of the beam is given by

$$y(x) = \frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2),$$

where w is weight of the beam per unit length, E and I are the Young's modulus and moment of inertia of the beam, respectively.

(i) Prove that $\frac{dy}{dx} > 0$ for all $0 < x < L$.

(ii) Find the maximum deflection of the beam in terms of w, E, I and L. Justify your answer.

(d) Suppose you have a sample of $1,000$ cells which grow to $1,400$ the next day. Given that the daily birth rate B and the daily death rate D of the population are constants. Assuming the Malthus model for population, evaluate $B - D$.

Question 3 [20 marks]

(a) Consider the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \\ \frac{dy}{dt} \end{bmatrix} = A \begin{bmatrix} x \\ \\ y \end{bmatrix},$$

where A is a non-singular 2×2 matrix. Given that A has an eigenvalue i whose corresponding eigenvector is $\begin{bmatrix} 2 \\ 1-i \end{bmatrix}$.

(i) Find the general solution to the system of differential equations.

(ii) Find the equilibrium point (or points) and classify it (or them) as one of the six types discussed in class.

(b) A company has two interacting branches, A and B. Given that

(i) for each \$1 worth of product A produces, it requires \$0.30 worth of its own product and \$0.50 worth of the product B produces;

(ii) for every \$1 worth of product B produces, it requires \$0.30 worth of its own product and \$0.40 worth of the product A produces.

Suppose that A produces \$150 worth of its product and B produces \$250 of its product. What is the dollar value of both products available for external consumption?

(c) Consider the system of differential equations

$$\begin{bmatrix} \frac{d^2x}{dt^2} \\ \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} -75 & 25 \\ \\ 50 & -50 \end{bmatrix} \begin{bmatrix} x \\ \\ y \end{bmatrix}.$$

Find the scalars u, v, and α such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ue^{\alpha t} \\ ve^{\alpha t} \end{bmatrix}$$

is a solution to the system of differential equations.

Question 4 [20 marks]

(a) Given that the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

has eigenvalues 1 and -1. Find the third eigenvalue and its corresponding eigenvector.

(b) Let A, B, C be three matrices such that $AB = AC$. Is it true that $B = C$? Justify your answer with a proof if it is true and with a counterexample if it is false.

(c) Let T be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ such that

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}.$$

(i) Find the matrix B that represents T.

(ii) Find B^{30}. Express your answer as a single matrix.

(d) Find real numbers a and b such that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a solution to the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

END OF PAPER

TURN OVER FOR TABLE OF LAPLACE TRANSFORMS

Laplace transform is denoted by L.

(1) $L(1) = \dfrac{1}{s}, \qquad s > 0$

(2) $L(t^n) = \dfrac{n!}{s^{n+1}}, \qquad s > 0,\ n = 1, 2, 3, \cdots$

(3) $L(\sin kt) = \dfrac{k}{s^2 + k^2}, \qquad s > 0$

(4) $L(\cos kt) = \dfrac{s}{s^2 + k^2}, \qquad s > 0$

(5) $L(t \sin kt) = \dfrac{2ks}{(s^2 + k^2)^2}, \qquad s > 0$

(6) $L(t \cos kt) = \dfrac{s^2 - k^2}{(s^2 + k^2)^2}, \qquad s > 0$

(7) $L(e^{at}) = \dfrac{1}{s - a}, \qquad s > a$

(8) $L(e^{at} f(t)) = (Lf)(s - a), \qquad s > a$

(9) $L(u(t - a) f(t - a)) = e^{-as} (Lf)(s), \qquad a \geq 0$

where $u(r) = 0$ if $r < 0$, $u(r) = 1$ if $r \geq 0$.

(10) $L(u(t - a) f(t)) = e^{-as} (L\{f(t + a)\})(s), \qquad a \geq 0$

where $u(r) = 0$ if $r < 0$, $u(r) = 1$ if $r \geq 0$.

(11) $L(\delta(t - a)) = e^{-as}, \qquad a \geq 0$

where $\delta(t)$ is the Dirac function at $t = 0$.

(12) $L(f^{(n)}(t)) = s^n\,(Lf)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$

END OF TABLE