Question 1

a) Find a potential function for the gradient vector field

$$\vec{F} = e^x\hat{\imath} + \frac{z}{y}\hat{\jmath} + \ln y\,\hat{k},$$

and evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, *where* C *is given by the vector function* $\vec{r}(t) = t\hat{\imath} + (t^2+1)\hat{\jmath} + (t^3+2)\hat{k}$ *for* $0 \le t \le 1$.

a) $f_x = e^x \quad \Rightarrow \quad f = e^x + g(y,z)$

$f_y = \dfrac{z}{y} \quad \Rightarrow \quad f = z\ln y + h(x,z)$

$f_z = \ln y \quad \Rightarrow \quad f = z\ln y + k(x,y)$

$f(x,y,z) = e^x + z\ln y + c$

$\vec{r}(0) = \hat{\jmath} + 2\hat{k}, \qquad \vec{r}(1) = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = f(1,2,3) - f(0,1,2) = e + 3\ln 2 - 1$$

Question 3

Let $f(x,y) = x^2y - y^2 + 2\sqrt{y}$.

a) Find the domain of $f(x,y)$.

b) Find the maximum rate of change of $f(x,y)$ *at the point* $(2,1)$ *and the direction in which it occurs.*

c) Find a unit vector $\vec{u}$ *such that* $D_{\vec{u}}f(2,1) = -3$.

a) domain of $f(x,y)$: $\{(x,y): x\in\mathbb{R}, y \ge 0\}$.

b) Let $\vec{u} = a\hat{\imath} + b\hat{\jmath}$ be a unit vector.

$D_{\vec{u}}f(2,1) = f_x(2,1)\times a + f_y(2,1)\times b = (4\hat{\imath}+3\hat{\jmath})\cdot(a\hat{\imath}+b\hat{\jmath}) = |4\hat{\imath}+3\hat{\jmath}||\vec{u}|\cos\theta$

where θ is the angle between $4\hat{\imath}+3\hat{\jmath}$ and $\vec{u}$.

Since $|\vec{u}| = 1$ and the largest value of $\cos\theta$ is 1, the maximum rate of change is $|4\hat{\imath}+3\hat{\jmath}| = 5$. This maximum rate occurs when $\theta = 0$, this means that $\vec{u}$ is in the same direction as $4\hat{\imath}+3\hat{\jmath}$.

c) $\vec{u} = a\hat{\imath} + b\hat{\jmath}$

$D_{\vec{u}}f(2,1) = f_x(2,1)\times a + f_y(2,1)\times b \quad \Rightarrow \quad 4a + 3b = -3$

Since $\vec{u}$ is a unit vector, $a^2 + b^2 = 1$.

$\therefore (a, b) = (0, -1), \left(-\frac{24}{25}, \frac{7}{25}\right)$

<u>Question 4</u>

Let $\vec{F} = y\hat{\imath} + xz\hat{\jmath} + z^2\hat{k}$

a) Find $\nabla \times \vec{F}$.

b) Use Stokes' Theorem to evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ *where C is the triangle with vertices* $(-1,0,0), (0,2,0)$ *and* $(0,0,3)$, *oriented clockwise when viewed from above.*

a) $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & z^2 \end{vmatrix} = -x\hat{\imath} + (z-1)\hat{k}$

b) The plane with boundary C, $-6x + 3y + 2z = 6$.

$\vec{r}(u, v) = u\hat{\imath} + v\hat{\jmath} + \left(3 + 3u - \frac{3}{2}v\right)\hat{k}$

$-\vec{r}_u \times \vec{r}_v = 3\hat{\imath} - \frac{3}{2}\hat{\jmath} - \hat{k} \quad [correctly\ oriented]$

$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_R -6u + \frac{3}{2}v - 2\, dA = \int_{-1}^{0} \int_{0}^{2+2u} -6u + \frac{3}{2}v - 2\, du\, dv = 1$

<u>Question 5</u>

Let P be the plane given by $z = k$. *Assume that* $0 < k < 1$, *so that P intersects the unit sphere centered at the origin in some curve C at height k. Let S denote the part of the sphere lying above the plane P, which has boundary C.*

a) Find a parameterization for the curve C, and describe the projection of S onto the xy-plane.

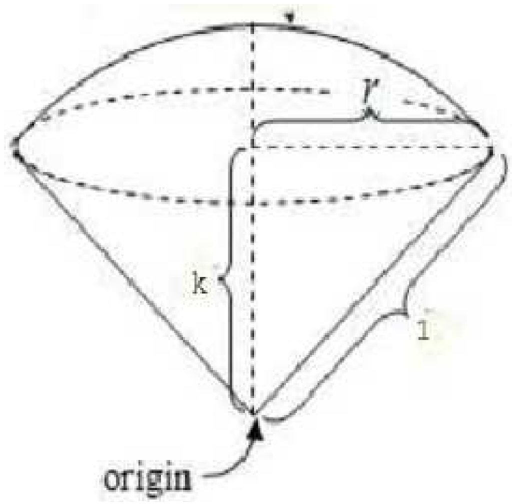
b) Write down and evaluate an integral which calculates the surface area of S in terms of k.

c) Find the value of k for which the surface area of S is equal to π.

a) The horizontal plane intersect the unit sphere at a circle. The radius r of this circle can be computed as $\sqrt{1-k^2}$ using Pythagoras Theorem as shown in the diagram.

$C : \vec{r}(t) = \sqrt{1-k^2}\cos t\,\hat{\imath} + \sqrt{1-k^2}\sin t\,\hat{\jmath} + k\hat{k}$

The projection of S onto the xy-plane is the circle of radius $\sqrt{1-k^2}$.



b)

$$f(x,y) = \sqrt{1-x^2-y^2}, \qquad f_x = -\frac{x}{\sqrt{1-x^2-y^2}}, \qquad f_y = -\frac{y}{\sqrt{1-x^2-y^2}}$$

Surface area,

$$\iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA = \int_0^{2\pi}\int_0^{\sqrt{1-k^2}} \frac{r}{\sqrt{1-r^2}}\, dr\, d\theta = 2\pi\int_0^{\sqrt{1-k^2}} \frac{r}{\sqrt{1-r^2}}\, dr = 2\pi(1-k)$$

c) $2\pi(1-k) = \pi$

$$\Rightarrow \quad k = \frac{1}{2}$$