Question 1

Use Green's Theorem to evaluate

$$\oint_C (1+10xy+y^2)dx + (6xy+5x^2)dy,$$

where C is the positively oriented triangle with vertices at $(0,0), (a,0)$ *and* $(0,a)$ *with* $a>0$.

$$\oint_C (1+10xy+y^2)dx + (6xy+5x^2)dy = \iint_D \frac{\partial}{\partial x}(6xy+5x^2) - \frac{\partial}{\partial y}(1+10xy+y^2)\,dA$$

$$= \iint_D (6y+10x) - (10x-2y)\,dA$$

$$= \iint_D 4y\,dA$$

$$= \int_0^a \int_0^{a-x} 4y\,dy\,dx$$

$$= \int_0^a 2(a-x)^2\,dx$$

$$= \frac{2}{3}a^3$$

Question 2

Let S be the surface $x^2+y^2=9,\ 0\le z\le 3$ *oriented with outward normal vector. Compute the surface integral* $\iint_S \vec{F}\cdot d\vec{S}$, *where* $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

S is a cylinder,

$\vec{r}(u,v) = 3\cos u\,\hat{\imath} + 3\sin u\,\hat{\jmath} + v\hat{k}, \qquad 0\le u\le 2\pi, \qquad 0\le v\le 3$

$\vec{r}_u = -3\sin u\,\hat{\imath} + 3\cos u\,\hat{\jmath}, \qquad \vec{r}_v = \hat{k}$

$\vec{r}_u \times \vec{r}_v = 3\cos u\,\hat{\imath} + 3\sin u\,\hat{\jmath}$, which is an outer normal vector.

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^3 \int_0^{2\pi} \left(3\cos u\,\hat{\imath} + 3\sin u\,\hat{\jmath} + v\hat{k}\right)\cdot(3\cos u\,\hat{\imath} + 3\sin u\,\hat{\jmath})du\,dv$$

$$= \int_0^3 \int_0^{2\pi} 9\,du\,dv$$

$$= 54\pi$$

Question 3

(i) *Let* $\vec{F}(x,y,z) = e^x\hat{\imath} + \cos y\,\hat{\jmath} + 2z\hat{k}$ *and* C *the curve of intersection of the plane* $2y + z = 5$ *and the cylinder* $x^2 + 4y^2 = 4$, *oriented counterclockwise when viewed from above.*
Use Stokes' Theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$.

(i)
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & \cos y & 2z \end{vmatrix} = 0$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = 0$$

Question 4

Use the method of separation of variables to find $u(x,y)$ *that satisfies the partial differential equation*

$$u_{xy} + \frac{\sin y}{x+2} u = 0$$

given that $u\left(2, \frac{\pi}{2}\right) = 10$ *and* $u\left(7, \frac{\pi}{2}\right) = 15$

$$u(x,y) = X(x)Y(y)$$

$$u_{xy} + \frac{\sin y}{x+2} u = 0$$

$$X'Y' + \frac{\sin y}{x+2} XY = 0$$

$$(x+2)\frac{X'}{X} = -\sin y \frac{Y}{Y'} = k$$

$$\frac{X'}{X} = \frac{k}{x+2} \quad \Rightarrow \quad \ln|X| = k\ln(x+2) + c \quad \Rightarrow \quad X = A(x+2)^k$$

$$\frac{Y'}{Y} = -\frac{1}{k}\sin y \quad \Rightarrow \quad \ln|Y| = \frac{1}{k}\cos y + d \quad \Rightarrow \quad Y = Be^{\frac{1}{k}\cos y}$$

$$u(x,y) = C(x+2)^k e^{\frac{1}{k}\cos y}$$

$$u\left(2, \frac{\pi}{2}\right) = C4^k = 10, \qquad u\left(7, \frac{\pi}{2}\right) = C9^k = 15 \quad \Rightarrow \quad C = 5, \qquad k = \frac{1}{2}$$

$$\therefore u(x,y) = 5\sqrt{x+2}\,e^{2\cos y}$$