Question 1 [20 marks]

(a) Solve the following initial value problem for $0 \le x < \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} + 9y = \sec 3x. \qquad y(0) = 2,\ \frac{dy}{dx}(0) = 3.$$

(b) A school of salmon, consisting initially of 250 fish, living off the Norwegian coast grows according to the Malthus model with a per capita birth rate of 0.3 per year.

(i) Assuming that there are no other factors which affect the population growth, predict the estimated population of salmon in three years' time.

(ii) Suppose that at time $t = 0$, a group of sharks establishes residence in the same area and begin attacking the salmon. The rate at which the salmon are killed is $0.001N(t)^2$, where $N(t)$ is the population of the salmon at time t. Moreover, due to the presence of predators, 20 salmon leave the area yearly. Write down a model for the population of salmon based on these assumptions.

(iii) Based on your model in part(ii), in the long run, would there be any more salmon living in this area? If so, find the estimated population.

a) $\frac{d^2y}{dx^2} + 9y = \sec 3x$

$auxilary\ equation, \qquad \lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda = \pm 3i$

$y_h = A\cos 3x + B\sin 3x$

$y_p = u\cos 3x + v\sin 3x$

$$\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$u = -\frac{1}{3}\int \sin 3x \sec 3x\, dx = \frac{1}{9}\ln|\cos 3x|, \qquad v = \frac{1}{3}\int \cos 3x \sec 3x\, dx = \frac{1}{3}x$$

$$y = y_h + y_p = A\cos 3x + B\sin 3x + \frac{1}{9}\cos 3x\,(\ln|\cos 3x|) + \frac{1}{3}x\sin 3x$$

$y(0) = 0, \qquad y'(0) = 3 \quad \Rightarrow \quad A = 2, \qquad B = 1$

$$\therefore y = \left(2 + \frac{1}{9}\ln|\cos 3x|\right)\cos 3x + \left(1 + \frac{1}{3}x\right)\sin 3x$$

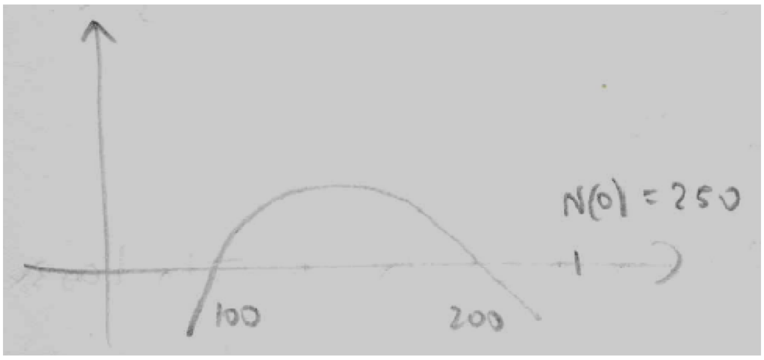
b) $N(t) = \widehat{N}e^{0.3t} = 250e^{0.3t}$

i) $N(3) = 250e^{0.9} \approx 615$

ii) $\frac{dN}{dt} = 0.3N - 0.001N^2 - 20$

iii) $\frac{dN}{dt} = -0.001(N^2 - 300N + 20000) = -0.001(N - 100)(N - 200)$

Equilibrium at 100 (unstable), 200(stable).



In the long run, $N(t) \to 200$ the stable equilibrium.

Question 2 [15 marks]

(a) Find the inverse Laplace transform of $\dfrac{3s}{(s+1)^4}$.

(b) An undamped mass spring oscillator system of mass 1kg is pulled down 2 units from the equilibrium and released from rest. You measured the oscillations and discovered that the period is 2 seconds. 10 seconds after the mass was pulled down, the mass is dealt an instantaneous blow that imparts 5 units of impulse in the upward direction. Taking downwards as positive, find the position of the mass 12.5 seconds after the mass was pulled down.

a) $\frac{3s}{(s+1)^4} = \frac{3(s+1)-3}{(s+1)^4} = \frac{3}{(s+1)^3} - \frac{3}{(s+1)^4}$

S-shifting, $L^{-1}\left[\frac{3s}{(s+1)^4}\right] = \frac{3}{2}t^2e^{-t} - \frac{1}{2}t^3e^{-t} = \frac{1}{2}t^2(3-t)e^{-t}$

b) $x + kx = 0 \quad \Rightarrow \quad \ddot{x} = -\omega^2 x, \quad \omega^2 = k$

$$Period = \frac{2\pi}{\sqrt{k}} = 2 \quad \Rightarrow \quad k = \pi^2$$

$\ddot{x} + \pi^2 x = -5\delta(t-10), \qquad x(0) = 2, \qquad \dot{x}(0) = 0$

Apply Laplace Transform,

$s^2X - sx(0) - \dot{x}(0) + \pi^2 X = -5e^{-10s}$

$$X = -\frac{5e^{-10s}}{s^2+\pi^2} + \frac{2s}{s^2+\pi^2}$$

t-shifting,

$$x(t) = -\frac{5}{\pi}u(t-10)\sin(t-10)\pi + 2\cos\pi t$$

$At\ t = 12.5, \qquad u(t-10) = 1$

$$\therefore x(12.5) = -\frac{5}{\pi}\sin 2.5\pi + 2\cos 12.5\pi = -\frac{5}{\pi}$$

Question 3 [10 marks]

Each year employees at a company are given the option of donating to a local charity as part of a payroll deduction plan. In general, 80 percent of the employees enrolled in the plan in any one year will choose to sign up again the following year, and 30 percent of the unenrolled will choose to enroll the following year.

(i) Write down the stochastic matrix M that describes this behaviour.

(ii) Compute the eigenvalues and corresponding eigenvectors of M.

(iii) Diagonalize M by writing it as a product PDP^{-1}, where D is a diagonal matrix.

(iv) Calculate the percentage of employees that would be enrolled in the long run.

i) $$M = \begin{array}{cc} \begin{array}{cc} \textit{enroll} & \textit{unenroll} \end{array} & \\ \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} & \begin{array}{c} \textit{enroll} \\ \textit{unenroll} \end{array} \end{array}$$

ii) $$\det(M - \lambda I) = \begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 0.5) = 0$$

$\lambda = 1$,

$$\begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\lambda = 0.5$,

$$\begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

iii) $$P = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{-3-2}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

$$M = \frac{1}{5}\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

iv) In the long run, 60% will be enrolled.

$$M^k = PD^kP^{-1} \to P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$$

Question 4 [15 marks]

A cantilevered beam of length L, made up of an extremely strong and light material, is horizontal at the end where it is attached to a wall. It carries a point load of P Newtons at the point $x = L$ and a point load of $2P$ Newtons at $x = \frac{L}{2}$. We assume that the weight of the beam is negligible compared to P. The deflection of the beam is described by the equation,

$$\frac{d^4y}{dx^4} = \frac{w(x)}{EI},$$

where E and I are constants.

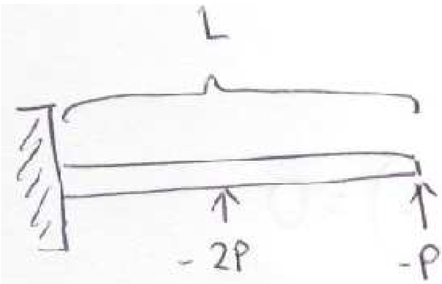
(i) Write down the load function $w(x)$.

(ii) Use Laplace transform to find the solution for $y(x)$, with the following initial conditions, $\frac{d^2y}{dx^2}(0) = -\frac{2PL}{EI}$, $\frac{d^3y}{dx^3}(0) = \frac{3P}{EI}$.

(iii) Show that the deflection at $x = \frac{L}{4}$ is

$$y\left(\frac{L}{4}\right) = -\frac{7PL^3}{128EI}.$$

(iv) If $L = 4$ metres and $P = 3$ Newtons, calculate the deflection at $x = \frac{3L}{4}$, leaving your answer in terms of E and I.



i) $w(x) = -2P\delta\left(x - \frac{L}{2}\right) - P\delta(x - L)$

ii) Apply Laplace transform with $y(0) = 0, y'(0) = 0,$

$$y''(0) = -\frac{2PL}{EI}, \qquad y'''(0) = \frac{3P}{EI}$$

$$s^4Y - s^3 \cdot 0 - s^2 \cdot 0 - s\left(-\frac{2PL}{EI}\right) - \frac{3P}{EI} = -\frac{2P}{EI}e^{-\frac{L}{2}s} - \frac{P}{EI}e^{-Ls}$$

$$Y = \frac{P}{EI}\left(\frac{3}{s^4} - \frac{2L}{s^3} - \frac{2}{s^4}e^{-\frac{L}{2}s} - \frac{P}{s^4}e^{-Ls}\right)$$

$$y(x) = \frac{P}{EI}\left(\frac{1}{2}x^3 - Lx^2 - \frac{1}{3}\left(x - \frac{L}{2}\right)^3 u\left(x - \frac{L}{2}\right) - \frac{1}{6}(x - L)^3 u(x - L)\right)$$

iii) $x = \frac{1}{4}$, $u\left(x - \frac{L}{2}\right) = u(x - L) = 0$

$$y\left(\frac{L}{4}\right)=\frac{P}{EI}\left[\frac{1}{2}\left(\frac{1}{4}\right)^3-L\left(\frac{L}{4}\right)^2\right]=-\frac{7PL^3}{128EI}$$

iv) $L=4,\ P=3,\ x=\frac{3L}{4}=3,\ u(x-L)=0,\ u\left(x-\frac{L}{2}\right)=1$

$$y(3)=\frac{3}{EI}\left[\frac{1}{2}\cdot 3^3-4\cdot 3^2-\frac{1}{3}(3-2)^2\right]=-\frac{137}{2EI}$$

Question 5 [20 marks]

(a) Solve the following initial value problem,

$$\frac{dy}{dx}+y\cot x=y^3\sin^3 x, \qquad y\left(\frac{\pi}{2}\right)=1.$$

(b) A small country's economy is divided into three sectors: Electronics (E), Solar energy production (S), and Pharmaceuticals (P). Using the Leontief input and output model, the technology matrix of the internal consumption is

Output \ Input	E	S	P
E	\$0.40	\$0.30	\$0.60
S	\$0.20	\$0.00	\$0.20
P	\$0.30	\$0.30	\$0.40

Suppose the export demands are as follows: Electronics: \$30 million, Solar energy: \$10 million and Pharmaceuticals: \$5 million. Find the total output demand for each of the three sectors.

a) Bernoulli form,

$$\frac{dy}{dx}+y\cot x=y^2\sin^3 x, \qquad y\left(\frac{\pi}{2}\right)=1$$

$$v=\frac{1}{y^2}\quad\Rightarrow\quad v'=-\frac{2y'}{y^3}$$

$$v'-2v\cot x=-2\sin^3 x$$

$$e^{-2\int\frac{\cos x}{\sin x}dx}=e^{-2\ln\sin x}=\csc^2 x$$

$$\frac{v}{\sin^2 x}=\int -2\sin x\,dx=2\cos x+c$$

$$\frac{1}{y^2}=v=2\cos x\sin^2 x+c\sin^2 x$$

$$x=\frac{\pi}{2},\qquad c=1$$

$$\therefore y = \frac{1}{\sin x\sqrt{2\cos x + 1}}$$

b) $(I - T)\begin{pmatrix} E \\ S \\ P \end{pmatrix} = \begin{pmatrix} 30 \\ 10 \\ 5 \end{pmatrix}$ *where*

$$I - T = \begin{pmatrix} 0.6 & -0.3 & -0.6 \\ -0.2 & 1 & -0.2 \\ -0.3 & -0.3 & 0.6 \end{pmatrix}$$

$$\det(I - T) = 0.6(0.54) + 0.3(-0.18) - 0.6(0.36) = 0.054$$

$$(I - T)^{-1} = \frac{1}{0.054}\begin{pmatrix} 0.54 & 0.18 & 0.36 \\ 0.36 & 0.18 & 0.27 \\ 0.66 & 0.24 & 0.54 \end{pmatrix}^T = \begin{pmatrix} 10 & \frac{40}{6} & \frac{110}{9} \\ \frac{10}{3} & \frac{10}{3} & \frac{40}{9} \\ \frac{40}{6} & 5 & 10 \end{pmatrix}$$

$$\therefore \begin{pmatrix} E \\ S \\ P \end{pmatrix} = (I - T)^{-1}\begin{pmatrix} 30 \\ 10 \\ 5 \end{pmatrix} \approx \begin{pmatrix} 427.8 \\ 155.6 \\ 300 \end{pmatrix}$$

Question 6 [20 marks]

(a) Let

$$X = \begin{bmatrix} 5 & 2 \\ -13 & -5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}.$$

(i) Classify the phase diagram of a system of differential equations associated with the matrix X.

(ii) Using X and A as given, find a matrix B such that $X = ABA^{-1}$.

(iii) Hence or otherwise, compute the matrix exponential of $e^{\theta X}$ as a matrix in terms of the variable θ.

(b) Consider two tanks, each containing 100 litres of water. Both tanks initially contain some amount of a salt dissolved in the water. Pure water is poured into tanks A and B at a constant rate of 1 litre per minute for each tank. The thoroughly mixed solution from tank A is constantly pumped into tank B at a rate of 1 litre per minute while the solution from tank B is pumped back to tank A at a rate of 2 litres per minute. The solution in tank A is also pumped out and discarded at a rate of 2 litres per minute.

Model this problem with a system of differential equations. Classify and sketch the phase diagram associated with this linear system, indicating clearly the eigenvectors. Suppose at some time t, the amount of salt in tank A is more than that in tank B. What can you conclude about the concentration of salt in both tanks.

a)

i) $X = \begin{pmatrix} 5 & 2 \\ -13 & -5 \end{pmatrix}$

$Tr\ X = 0, \qquad \det X = 1, \qquad Tr^2 X - 4\det X < 0 \quad \Rightarrow \quad Centre$

ii) $A = \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}$

$X = ABA^{-1} \quad \Rightarrow \quad B = A^{-1}XA$

$B = \begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ -13 & -5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

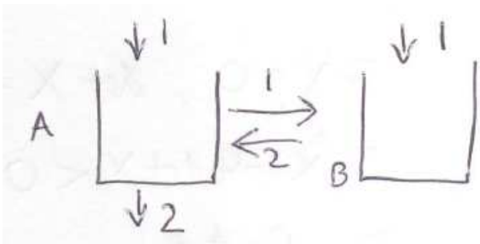
iii) $e^{\theta X} = e^{\theta ABA^{-1}}$

$= Ae^{\theta B}A^{-1}$

$= \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} \cos\theta + 5\sin\theta & 2\sin\theta \\ -13\sin\theta & \cos\theta - 5\sin\theta \end{pmatrix}$

b) Let X_A, X_B be the amount of salt in tanks A & B.



$\frac{d}{dt}\begin{pmatrix} X_A \\ X_B \end{pmatrix} = \frac{1}{100}\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} X_A \\ X_B \end{pmatrix}$

For simplicity, we ignore the factor $\frac{1}{100}$, the phase diagrams remain the same.

$M = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}, \qquad Tr\ M = -5, \qquad \det M = 4$

$Tr^2\ M - 4\det M > 0 \quad \Rightarrow \quad nodal\ sink$

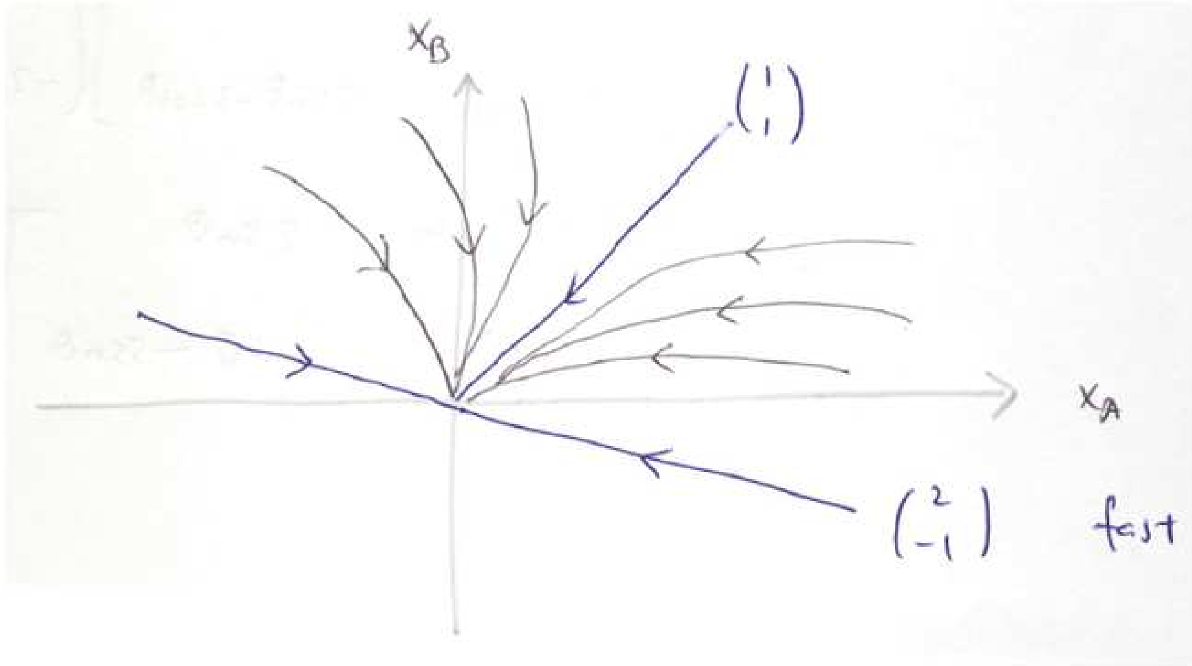
$\begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+1)(\lambda+4) = 0 \quad \Rightarrow \quad \lambda = -1, -4$

$\lambda = -1,$

$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad eigenvector\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -1,$

$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad eigenvector\ \begin{pmatrix} 2 \\ -1 \end{pmatrix}$



$X_A(t) > X_B(t)$

Since trajectories never cross, $X_A(t) > X_B(t)$ for $\forall t$ until $X_A = X_B = 0$.