<u>Question 3</u>

Let $\vec{F} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$.

(i) *Find* $\nabla \times \vec{F}$.

(ii) *Use Stokes' Theorem to evaluate the line integral* $\int_C \vec{F} \cdot d\vec{r}$ *where C is the triangle with vertices* $(1,0,0), (0,1,0), (0,0,1)$ *oriented counter-clockwise when viewed from above.*

(i) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = -y\hat{\imath} - z\hat{\jmath} - x\hat{k}$$

(ii) Stokes' Theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

Equation of S, $z = 1 - x - y \quad \Rightarrow \quad \vec{r}(u,v) = u\hat{\imath} + v\hat{\jmath} + (1-u-v)\hat{k},\ (u,v) \in D$

where D is the projection of the portion of S in first octant onto the xy-plane.

$0 \le v \le 1 - u, \qquad 0 \le u \le 1$

$\vec{r}_u \times \vec{r}_v = \hat{\imath} + \hat{\jmath} + \hat{k}$, normal vector points upwards, and thus agrees with the orientation of the boundary curve C (counter-clockwise).

$\nabla \times \vec{F}(\vec{r}) = -v\hat{\imath} - (1-u-v)\hat{\jmath} - u\hat{k}$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_D \left[-v\hat{\imath} - (1-u-v)\hat{\jmath} - u\hat{k}\right] \cdot \left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) dA$$

$$= \int_0^1 \int_0^{1-u} -1 \, dv \, du$$

$$= \frac{1}{2}$$