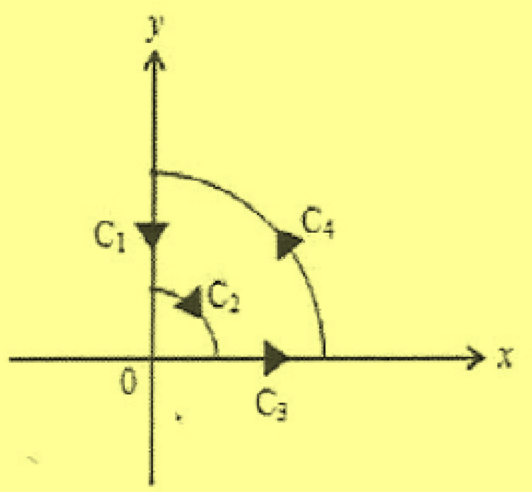
<u>Question 4</u>

A closed curve with positive orientation is made up of 4 curves C_1, C_2, C_3 and C_4 as shown in the diagram below.



C_2 is the portion of the unit circle $x^2 + y^2 = 1$ that lies in the 1st quadrant. C_4 is the portion of the circle $x^2 + y^2 = 9$ that lies in the 1st quadrant. Evaluate the following line integrals:

(a)

$$\oint_{C_1+C_2+C_3+C_4} \left(-\frac{y}{x^2+y^2} + y^2\right) dx + \left(\frac{x}{x^2+y^2} - x^2\right) dy,$$

(b)

$$\int_{C_1+C_2+C_3} \left(-\frac{y}{x^2+y^2} + y^2\right) dx + \left(\frac{x}{x^2+y^2} - x^2\right) dy$$

(a) By Green's theorem,

$$\begin{aligned}
\oint_C \left(-\frac{y}{x^2+y^2} + y^2\right) dx + \left(\frac{x}{x^2+y^2} - x^2\right) dy &= \iint_D \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - x^2\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2} + y^2\right) \\
&= \iint_D \left(\frac{y^2-x^2}{(x^2+y^2)^2} - 2x\right) - \left(\frac{y^2-x^2}{(x^2+y^2)^2} + 2y\right) dA \\
&= \iint_D -2(x+y)\, dA \\
&= \int_0^{\frac{\pi}{2}} \int_1^3 -2r(r\cos\theta + r\sin\theta)\, dr\, d\theta \\
&= -\frac{104}{3}
\end{aligned}$$

(b)

$$\int_{C_1+C_2+C_3}\left(-\frac{y}{x^2+y^2}+y^2\right)dx+\left(\frac{x}{x^2+y^2}-x^2\right)dy$$
$$=\oint_{C_1+C_2+C_3+C_4}\left(-\frac{y}{x^2+y^2}+y^2\right)dx+\left(\frac{x}{x^2+y^2}-x^2\right)dy$$
$$-\int_{C_4}\left(-\frac{y}{x^2+y^2}+y^2\right)dx+\left(\frac{x}{x^2+y^2}-x^2\right)dy$$

$C_4 : \vec{r}(\theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath}, \qquad 0 \le \theta \le \frac{\pi}{2}$

$\vec{r}'(\theta) = -3\sin\theta\,\hat{\imath} + 3\cos\theta\,\hat{\jmath}$

$$\int_{C_4}\left(-\frac{y}{x^2+y^2}+y^2\right)dx+\left(\frac{x}{x^2+y^2}-x^2\right)dy=\int_{C_4}\vec{F}\cdot d\vec{r}$$
$$=\int_0^{\frac{\pi}{2}}\vec{F}[\vec{r}(\theta)]\cdot\vec{r}'(\theta)d\theta$$
$$=\int_0^{\frac{\pi}{2}}\sin^2\theta-27\sin^3\theta+\cos^2\theta-27\cos^3\theta\,d\theta$$
$$=\frac{\pi}{2}-36$$

$$\therefore\int_{C_1+C_2+C_3}\left(-\frac{y}{x^2+y^2}+y^2\right)dx+\left(\frac{x}{x^2+y^2}-x^2\right)dy=-\frac{104}{3}-\left(\frac{\pi}{2}-36\right)=\frac{4}{3}-\frac{\pi}{2}$$

__Question 5__

Let S be the portion of the unit sphere $x^2+y^2+z^2=1$ in the first octant and let C be the boundary of S. The orientation of C is counterclockwise when looking down at the surface S. Find a vector field $\vec{G}(x,y,z)$ such that

$$\oint_C x^2dx+2xy\,dy+xz\,dz=\iint_S\vec{G}(x,y,z)\cdot d\vec{S}$$

and evaluate the surface integral $\iint_S\vec{G}(x,y,z)\cdot d\vec{S}$ directly.

By Stokes' theorem, $\vec{G}=\nabla\times\vec{F}$,

$\vec{G}=-z\hat{\jmath}+2y\hat{k}$

$\vec{r}(u,v)=\sin u\cos v\,\hat{\imath}+\sin u\sin v\,\hat{\jmath}+\cos u\,\hat{k}, \qquad 0\le u\le\frac{\pi}{2}, \qquad 0\le v\le\frac{\pi}{2}$

$\vec{r}_u\times\vec{r}_v=\sin^2 u\cos v\,\hat{\imath}+\sin^2 u\sin v\,\hat{\jmath}+\sin u\cos u\,\hat{k}$

The k-component is non-negative, so this is the upward-pointing normal vector, and this gives the correct orientation on S.

$$\iint_S \vec{G}(x,y,z)\cdot d\vec{S} = \iint_D \vec{G}[\vec{r}(u,v)]\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

$$\vec{G}[\vec{r}(u,v)] = -\cos u\,\hat{j} + 2\sin u \sin v\,\hat{k}$$

$$\therefore \iint_D \vec{G}[\vec{r}(u,v)]\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(-\cos u\,\hat{j} + 2\sin u\sin v\,\hat{k}\right)\cdot\left(\sin^2 u\cos v\,\hat{\imath} + \sin^2 u \sin v\,\hat{j} + \sin u\cos u\,\hat{k}\right)du\,dv$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2 u\cos u\sin v\,du\,dv$$

$$= \frac{1}{3}$$