Question 1

a) Let $y(x)$ be the solution of the initial value problem

$$y\frac{dy}{dx} = x^2, \qquad y > 0, \qquad y(0) = \sqrt{7}$$

Find the value of $y(3)$.

b) Glucose is added intravenously to the bloodstream at a rate of 5 units per minute, and the body removes glucose from the bloodstream at a rate proportional to the amount of glucose present. Initially there are 60 units of glucose in the bloodstream. After 2 minutes, there are w units of glucose in the bloodstream. If after a very long time, there are 10 units of glucose in the bloodstream, what is the value of w?

a) $y\,dy = x^2 dx$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + c$$

$$\frac{1}{2}\left(\sqrt{7}\right)^2 = \frac{1}{3}(0)^3 + c \quad \Rightarrow \quad c = \frac{7}{2}$$

$$y^2 = \frac{2}{3}x^3 + 7$$

$$\therefore y(3) = \sqrt{\frac{2}{3}(3^3) + 7} = 5$$

b) $\frac{dx}{dt} = 5 - kx = -k\left(x - \frac{5}{k}\right), \; k > 0$

$$\ln\left|x - \frac{5}{k}\right| = -kt + c \quad \Rightarrow \quad x = \frac{5}{k} + Ae^{-kt}$$

$$x(0) = 60 \quad \Rightarrow \quad A = 60 - \frac{5}{k}$$

$$x = \frac{5}{k} + \left(60 - \frac{5}{k}\right)e^{-kt}$$

$$\lim_{x\to\infty} x = 10 \quad \Rightarrow \quad k = \frac{1}{2}$$

$$x = 10 + 50e^{-\frac{1}{2}t}$$

$$\therefore x(2) = 10 + 50e^{-1} \approx 28.39$$

Question 2

a) Let $y(x)$ be a solution of the initial value problem

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x^2}y^3, \qquad x>0, \qquad y^2(1)=\frac{5}{7}$$

Find the exact value of $y^2(2)$.

b) The growth of rabbits in your rabbit farm followed a logistic population model with a birth rate per capita of 10 rabbits per rabbit per year. You observed that their number had approached to a logistic equilibrium population of 2 500 rabbits. One day your friend Dr Good visited your farm and suggested that you try to mix some of his latest invention of Vitamin X into your rabbit feed to boost the reproduction rate. You followed his suggestion and after a long period of time, observed that the rabbit population had reached a new logistic equilibrium of 3 000 rabbits. If the new rabbit birth rate per capita after Vitamin X was introduced was B rabbits per rabbit per year, what is the value of B?

a) $z=y^{1-3}=\frac{1}{y^2} \quad \Rightarrow \quad dz=-\frac{2}{y^3}dy$

$$-\frac{1}{2}y^3\frac{dz}{dx}+\frac{2}{x}y=\frac{1}{x^2}y^3$$

$$\frac{dz}{dx}-\frac{4}{x}z=-\frac{2}{x^2}$$

$$e^{\int -\frac{4}{x}dx}=e^{-4\ln x}=\frac{1}{x^4}$$

$$z=x^4\int\frac{1}{x^4}\left(-\frac{2}{x^2}\right)dx=x^4\left(\frac{2}{5x^5}+c\right)$$

$$\frac{1}{y^2}=\frac{2}{5x}+cx^4$$

$$y^2(1)=\frac{5}{7} \quad \Rightarrow \quad c=1$$

$$y^2=\frac{1}{\frac{2}{5x}+x^4}$$

$$\therefore y^2(2)=\frac{1}{\frac{1}{5}+16}=\frac{5}{81}$$

b) $\frac{10}{s}=2500 \quad \Rightarrow \quad s=\frac{10}{2500}=\frac{1}{250}$

$$\therefore \frac{B}{s}=3000 \quad \Rightarrow B=\frac{3000}{250}=12$$

Question 3

a) An oil drum in the form of a 4 feet long right circular cylinder is in equilibrium when it floats upright with its axis vertical and half submerged in a lake. The oil drum is then further submerged so that it extends 1 foot above water, and then released from rest at time $t = 0$*. Assume that there is no friction between the sides of the drum and the water in the lake and that the gravitational constant equals to* $32ft/s^2$ *, find the exact value in feet, of the length of the part of the drum that is above water at time* $t = \frac{\pi}{12}s$*.*

b) The population x of a rare species of monkey is governed by the equation

$$\frac{dx}{dt} = -\frac{11}{8}x(20-x)\left(1-\frac{1}{50}x\right)$$

Initially there are q monkeys.

(i) If $q = 19$*, what will the monkey population eventually be, after a very long time?*

(ii) If $q = 157$*, what will the monkey population eventually be, after a very long time?*

a) Let x be the length above water at time t, ρ be the density of water and A the cross sectional area.

$mg = \rho 2Ag \quad \Rightarrow \quad m = 2\rho A$

$m\ddot{x} = \rho(4-x)Ag - mg = 4\rho Ag - x\rho Ag - 2\rho Ag$

$2\rho A\ddot{x} = 2\rho Ag - x\rho Ag \quad \Rightarrow \quad \ddot{x} = g - \frac{x}{2}g$

$\ddot{x} + 16x = 32$

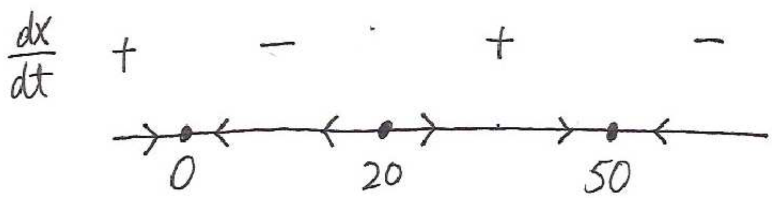
$x = A\cos 4t + B\sin 4t + 2$

$x(0) = 1, \dot{x}(0) = 0 \quad \Rightarrow \quad A = -1, B = 0$

$x = 2 - \cos 4t$

$\therefore x\left(\frac{\pi}{12}\right) = 2 - \cos\frac{\pi}{3} = \frac{3}{2}$

b) i) 0; ii) 50



Question 4

a) A cantilevered beam of length L has a weight per unit length given by

$$\frac{2ax}{L},$$

where a is constant and x measures distance from the point of attachment. It is horizontal at the end where it is attached to a wall. Find the maximum deflection at $x = L$*.*

b) Use Laplace transform to solve the following initial value problem

$$y'' - y' = te^t + 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

a) $\frac{d^4y}{dx^4} = -\frac{2ax}{EIL} \quad \Rightarrow \quad \frac{d^3y}{dx^3} = -\frac{ax^2}{EIL} + c_1$

$$0 = -\frac{aL^2}{EIL} + c_1 \quad \Rightarrow \quad \frac{d^3y}{dx^3} = -\frac{ax^2}{EIL} + \frac{aL}{EI}$$

$$\frac{d^2y}{dx^2} = -\frac{ax^3}{3EIL} + \frac{aL}{EI}x + c_2$$

$$0 = -\frac{aL^3}{3EIL} + \frac{aL^2}{EI} + c_2 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{ax^3}{3EIL} + \frac{aLx}{EI} - \frac{2aL^2}{3EI}$$

$$\frac{dy}{dx}(0) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{ax^4}{12EIL} + \frac{aLx^2}{2EI} - \frac{2aL^2x}{3EI}$$

$$y(0) = 0 \quad \Rightarrow \quad y = -\frac{ax^5}{60EIL} + \frac{aLx^3}{6EI} - \frac{aL^2x^2}{3EI}$$

$$\therefore \Delta = -\frac{aL^4}{60EI} + \frac{aL^4}{6EI} - \frac{aL^4}{3EI} = -\frac{11}{60}\left(\frac{aL^4}{EI}\right)$$

b) $s^2L(y) - s - sL(y) + 1 = \frac{1}{(s-1)^2} + \frac{1}{s}$

$$s(s-1)L(y) = s - 1 + \frac{1}{(s-1)^2} + \frac{1}{s}$$

$$L(y) = \frac{1}{s} + \frac{1}{s(s-1)^3} + \frac{1}{s^2(s-1)}$$

$$= \frac{1}{s} + \left[-\frac{1}{s} + \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}\right] + \left[-\frac{1}{s} - \frac{1}{s^2} + \frac{2}{s-1}\right]$$

$$= -\frac{1}{s} - \frac{1}{s^2} + \frac{2}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$\therefore y = -1 - t + 2e^t - te^t + \frac{1}{2}t^2e^t$$

Question 5

a) *In an RLC circuit of inductance **L** henrys, resistance **R** ohms, capacitance **C** farads and voltage **V** volts, it is known that the electric current I satisfies the equation*

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(u)du = V.$$

At time $t = 0s$, seeing that there is no voltage applied to the circuit and that $I = 0$ at that time, you immediately turn the switch on and off, firing a short burst of voltage into it. You observe that the resulting current at $t > 0$ satisfies

$$I(t) = e^{-3t}\cos t - 3e^{-3t}\sin t$$

If $L = \frac{1}{2}$, find the value of R and C.

b) *The growth of a certain fish population $N(t)$ is governed by*

$$\frac{dN}{dt} = 0.01N,$$

where time is measured in days. Initially at day 0, $N(0) = 1000$ tons. Harvesting is allowed for a 30 days period beginning with day 0 at the rate of 20 tons per day. This can be modeled by the equation

$$\frac{dN}{dt} = 0.01N - 20[1 - u(t - 30)],$$

where u is the unit step function. Find the amount of fish in tons

i) *at time $t = 25$;*

ii) *at time $t = 50$.*

a) $Ls\mathcal{L}(I) + R\mathcal{L}(I) + \frac{1}{sC}\mathcal{L}(I) = \mathcal{L}[A\delta(t)] = A$

$$\mathcal{L}(I) = \frac{A}{Ls + R + \frac{1}{sC}}$$

$$= \frac{As}{\frac{1}{2}s^2 + Rs + \frac{1}{C}}$$

$$= \mathcal{L}(e^{-3t}\cos t - 3e^{-3t}\sin t)$$

$$= \frac{s+3}{(s+3)^2+1} - 3\frac{1}{(s+3)^2+1}$$

$$= \frac{s}{s^2 + 6s + 10}$$

$$= \frac{As}{As^2 + 6As + 10A}$$

$$A = \frac{1}{2}, \qquad R = 6A = 3, \qquad \frac{1}{C} = 10A = 5 \quad \Rightarrow \quad C = \frac{1}{5}$$

b) $sL(N) - 1000 = 0.01L(N) - 20\left(\frac{1}{s} - \frac{e^{-30s}}{s}\right)$

$$L(N) = \frac{1000}{s - 0.01} - \frac{20}{s(s-0.01)} + \frac{20}{s(s-0.01)}e^{-30s}$$

$$= \frac{1000}{s-0.01} - 2000\left(\frac{1}{s-0.01} - \frac{1}{s}\right) + 2000\left(\frac{1}{s-0.01} - \frac{1}{s}\right)e^{-30s}$$

$$N = 1000e^{0.01t} - 2000e^{0.01t} + 2000 + 2000\left[e^{0.01(t-30)} - 1\right]u(t-30)$$

$$= 2000 - 1000e^{0.01t} + 2000\left[e^{0.01(t-30)} - 1\right]u(t-30)$$

$i)$ $N(25) = 2000 - 1000e^{0.25} \approx 715.97$

$ii)$ $N(50) = 2000 - 1000e^{0.5} + 2000(e^{0.2} - 1) \approx 794.08$

Question 6

a) The probability that a smoker will quit smoking and become a non-smoker a year later is 30%. The probability a non-smoker will become a smoker a year later is 10%. We use the following transition matrix to represent this information:

$$\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix},$$

Mr Tan is currently a smoker. What is the probability that he becomes a non-smoker 3 years from now?

b) Let A denote the matrix $\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$. Find the exact expression of the matrix e^A.

a) $\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}^2 = \begin{pmatrix} 0.52 & 0.16 \\ 0.48 & 0.84 \end{pmatrix}$

$$\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}^3 = \begin{pmatrix} 0.52 & 0.16 \\ 0.48 & 0.84 \end{pmatrix}\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix} = \begin{pmatrix} * & * \\ 0.588 & * \end{pmatrix}$$

b) $\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \quad \lambda = -1, 3$$

$$\lambda = -1 \quad \Rightarrow \quad x + 2y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\lambda = 3 \quad \Rightarrow \quad x - 2y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

$$A = P\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}P^{-1}$$

$$\therefore e^A = \frac{1}{4}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e^3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e^{-1} + e^3) & -e^{-1} + e^3 \\ \frac{1}{4}(-e^{-1} + e^3) & \frac{1}{2}(e^{-1} + e^3) \end{pmatrix}$$

Question 7

a) Find the new coordinates of the point $(2,1)$ *if we shear 45˚ parallel to the x-axis, and then rotate 135˚ anticlockwise.*

b) Classify the following system of differential equations

$$\frac{dx}{dt} = -2x - 5y, \qquad \frac{dy}{dt} = x - 3y$$

and then carefully and clearly draw its phase plane diagram in the space below.

a) Shear 45°: $S = \begin{pmatrix} 1 & \tan 45° \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Rotate 135°: $R = \begin{pmatrix} \cos 135° & -\sin 135° \\ \sin 135° & \cos 135° \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

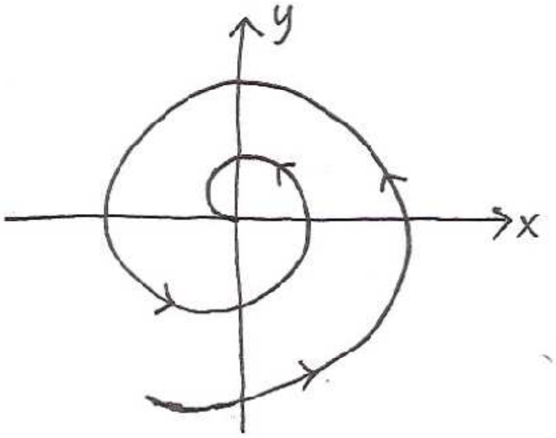
$$\therefore \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

b) $det = 6 + 5 = 11,\ Tr = -2 - 3 = -5$

$Tr^2 - 4det = 25 - 44 < 0$

∴ it is a Spiral Sink.

At $(\alpha, 0)$ *where* $\alpha > 0, \dfrac{dy}{dx} = \alpha > 0 \quad \Rightarrow \quad graph\ goes \uparrow$



Question 8

a) Solve the following system of differential equations:

$$\frac{dx}{dt} = -4x + 3y, \qquad \frac{dy}{dt} = -2x + y$$

with $x(0) = 4$ *and* $y(0) = 3$.

b) The orcs of Saruman have gone to war against the orcs of Sauron. Both sides use diseases against each other, as well as conventional weapons. Suppose that we model this situation using the system of ordinary differential equations

$$\frac{dx}{dt} = -4x - 3y, \qquad \frac{dy}{dt} = -x - 2y$$

where x and y denote the number of Saruman's orcs and Suron's orcs respectively at any time t. At time $t = 0$, *Saruman sends 15 000 orcs against an army of k Sauron orcs. What is the minimum positive integer N that k must exceed in order that all Saruman orcs are killed in the battle and some Sauron orcs will survive the battle?*

a) $\begin{vmatrix} -4-\lambda & 3 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = -2, -1$

$\lambda = -2 \quad \Rightarrow \quad -2x + 3y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda = -1 \quad \Rightarrow \quad -2x + 2y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + Be^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3Ae^{-2t} + Be^{-t} \\ 2Ae^{-2t} + Be^{-t} \end{pmatrix}$

$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3A + B \\ 2A + B \end{pmatrix} \quad \Rightarrow \quad A = 1, \qquad B = 1$

$\therefore x = 3e^{-2t} + e^{-t}, \qquad y = 2e^{-2t} + e^{-t}$

b) $\begin{vmatrix} -4-\lambda & -3 \\ -1 & -2-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 6\lambda + 5 \quad \Rightarrow \quad \lambda = -1, -5$

$\lambda = -5 \quad \Rightarrow \quad x - 3y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = -1 \quad \Rightarrow \quad x - 3y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$(15000, k)$ lies above $x - 3y = 0$

$\therefore 3k > 15000 \quad \Rightarrow \quad k > 5000$