Question 1

a) Let $y > 0$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$$

with the initial condition $y(0) = 1$. Find the value of $y(1)$.

b) An old object was dug up near NUS and you were asked to help to find out how old is this object. You carried out an experiment and found that the object contained 95% of the carbon-14 found in a similar present-day sample. You looked up a table from your chemistry book and found that the half-life of carbon-14 is 5730 years. Assume that carbon-14 decays at a rate proportional to the amount present, approximately how many years old is this object?

a) $2y\,dy = (3x^2 + 1)dx$

$$y^2 = x^3 + x + c$$

$$y(0) = 1 \quad \Rightarrow \quad c = 1$$

$$y^2 = x^3 + x + 1$$

$$y^2(1) = 3$$

$$\therefore y(1) = \sqrt{3}$$

b)

$$\frac{dx}{dt} = -kx \quad \Rightarrow \quad x = Ae^{-kt}$$

$$\frac{1}{2}A = Ae^{-5730k} \quad \Rightarrow \quad k = \frac{\ln 2}{5730}$$

$$x = Ae^{-\frac{\ln 2}{5730}t}$$

$$0.95A = Ae^{-\frac{\ln 2}{5730}t}$$

$$\therefore t = -\frac{5730 \ln 0.95}{\ln 2} \approx 424 \text{ } years$$

Question 2

a) Solve the differential equation $y'' + 4y = \cos^2 x - \sin^2 x$ with the initial conditions $y(0) = 1,\ y'(0) = 2$.

b) A particle moves along the x-axis in accordance with the equation of motion $\ddot{x} + 6\dot{x} - 16x = 0$. At $t = 0s$, the particle is at $x = 2m$ and moving to the left with a velocity of $10ms^{-1}$. When will the particle change direction and go to the right?

a) $y'' + 4y = \cos^2 x - \sin^2 x = \cos 2x$

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda = \pm 2i$$

$y = Re(z)$ where $z'' + 4z = e^{i2x}$

$z = Axe^{i2x}$

$z' = Ae^{i2x} + 2iAxe^{i2x}$

$z'' = 4iAe^{i2x} - 4Axe^{i2x}$

$4iAei2x = e^{i2x} \quad \Rightarrow \quad A = -\frac{i}{4}$

$z = -\frac{i}{4}x(\cos 2x + i\sin 2x) = \frac{1}{4}x\sin 2x - \frac{1}{4}ix\cos 2x$

$y = B\cos 2x + C\sin 2x + \frac{1}{4}x\sin 2x$

$y(0) = 1, y'(0) = 2 \quad \Rightarrow \quad B = 1, \qquad C = 1$

$\therefore y = \cos 2x + \sin 2x + \frac{1}{4}x\sin 2x$

b) $\lambda^2 + 6\lambda - 16 \quad \Rightarrow \quad \lambda = 2, -8$

$x = Ae^{2t} + Be^{-8t}$

$\dot{x} = 2Ae^{2t} - 8Be^{-8t}$

$x(0) = 2, \dot{x}(0) = -10 \quad \Rightarrow \quad A = \frac{3}{5}, \qquad B = \frac{7}{5}$

$\dot{x} = \frac{6}{5}e^{2t} - \frac{56}{5}e^{-8t}$

The particle will change direction when $\dot{x} = 0$,

$\frac{6}{5}e^{2t} - \frac{56}{5}e^{-8t} = 0 \quad \Rightarrow \quad e^{10t} = \frac{28}{3}$

$\therefore t = \frac{1}{10}\ln\frac{28}{3} \approx 0.22336$

Question 3

a) *A psychologist used the equation*

$$\frac{dP}{dt} = \frac{1}{1+t^2}(M - 2tP)$$

to model the performance of a certain student. Here P denotes the student's performance at any time $t \geq 0$ *and M denotes a positive constant. Assume that* $P = 0$ *at time* $t = 0$. *What is the value of t when P first reaches 40%?*

b) *A certain bird population has a birth rate of 10% per year. They had been protected by law for many years and attained a logistic equilibrium of 1000 000 birds. The government then allowed people to shoot E birds per year and after a long time, the population settled down to a new equilibrium of 68 000 birds. Find the value of E.*

a)

$$\frac{dP}{dt} + \frac{2t}{1+t^2}P = \frac{1}{1+t^2}M$$

$$e^{\int \frac{2t}{1+t^2}dt} = e^{\ln(1+t^2)} = 1 + t^2$$

$$P = \frac{1}{1+t^2}\int \frac{1+t^2}{1+t^2}M\,dt = \frac{1}{1+t^2}(Mt + c)$$

$$P(0) = 0 \quad \Rightarrow \quad c = 0$$

$$P = \frac{1}{1+t^2}Mt$$

$$\frac{P}{M} = \frac{2}{5} = \frac{t}{1+t^2}$$

$$2t^2 - 5t + 2 = 0 \quad \Rightarrow \quad t = \frac{1}{2}, 2$$

$$\therefore t = \frac{1}{2}$$

b)

$$B_\infty = \frac{B}{S} \quad \Rightarrow \quad S = \frac{1}{10000000}$$

$$\frac{dN}{dt} = BN - SN^2 - E$$

$$= \frac{1}{10}N - \frac{1}{1000000}N^2 - E$$

$$= -\frac{1}{1000000}(N^2 - 100000N + 1000000E)$$

Let $\beta_1 < \beta_2$ be 2 roots of $N^2 - 100000N + 1000000E$.

$\beta_2 = 68000$

$\beta_1 + \beta_2 = 100000 \quad \Rightarrow \quad \beta_1 = 32000$

$1000000E = \beta_1\beta_2 = 32000(68000)$

$\therefore E = 32(68) = 2176$

Question 4

a) A cantilevered beam of length L, made up of an extremely strong and light material, is horizontal at the end where it is attached to a wall, and carries a load of P Newtons at its end ($x = L$). Assuming the weight of the beam is negligible compared to P, the beam has a moment function given by $M(x) = -(L-x)P$. Find the maximum deflection at $x = L$.

b) Let $g(t)$ be defined as

$$g(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 4, & 2 \le t < 4 \\ 0, & t \ge 4 \end{cases}$$

Sketch $g(t)$ and compute its Laplace transform, $G(s)$, at $s = 4$.

a)

$$M(x) = EI\frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{\rho}{EI}(L-x)$$

$$\frac{dy}{dx} = -\frac{\rho}{EI}\left(Lx - \frac{1}{2}x^2\right) + c$$

$$\frac{dy}{dx}(0) = 0 \quad \Rightarrow \quad c = 0$$

$$y = -\frac{\rho}{EI}\left(\frac{L}{2}x^2 - \frac{1}{6}x^3\right) + d$$

$$y(0) = 0 \quad \Rightarrow \quad d = 0$$

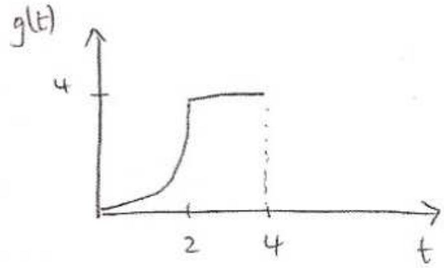
$$y(x) = -\frac{\rho}{EI}\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right)$$

$$\therefore y(L) = -\frac{\rho}{6EI}(3L^3 - L^3) = -\frac{\rho L^3}{3EI}$$

b) $g(t) = t^2[u(t) - u(t-2)] + 4[u(t-2) - u(t-4)]$

$= t^2u(t) - [(t-2)^2 + 4(t-2)]u(t-2) - 4u(t-4)$

$$L[g(t)] = G(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right) - \frac{4e^{-4s}}{s}$$



$$\therefore G(4) = \frac{2}{64} - e^{-8}\left(\frac{1}{32} + \frac{1}{4}\right) - e^{-16} \approx 0.0312$$

Question 5

a) Find the inverse Laplace transform of

$$\frac{1}{(s-1)(s^2+9)}$$

b) Solve the following initial value problem $y'' + 2\pi y' + 4\pi^2 y = \delta(t-1)$ *with initial conditions* $y(0) = 1$ *and* $y'(0) = -\pi$*. Evaluate* $y(2)$*.*

a)

$$\frac{1}{(s-1)(s^2+9)} = \frac{1}{10}\frac{1}{s-1} - \frac{1}{10}\frac{s}{s^2+9} - \frac{1}{10}\frac{s}{s^2+9}$$

$$\therefore L^{-1}\left[\frac{1}{(s-1)(s^2+9)}\right] = L^{-1}\left(\frac{1}{10}\frac{1}{s-1} - \frac{1}{10}\frac{s}{s^2+9} - \frac{1}{10}\frac{s}{s^2+9}\right)$$

$$= \frac{1}{10}e^t - \frac{1}{10}\cos 3t - \frac{1}{30}\sin 3t$$

b) $L[\delta(t-1)] = e^{-s}$

$$Y(s) = L(y) = \frac{s + \pi + e^{-s}}{(s+\pi)^2 + 3\pi^2}$$

$$y(t) = L^{-1}\left[\frac{s + \pi + e^{-s}}{(s+\pi)^2 + 3\pi^2}\right] = e^{-\pi t}\cos\sqrt{3}\pi t + \frac{1}{\sqrt{3}\pi}e^{-\pi(t-1)}\sin\left[\sqrt{3}\pi(t-1)\right]u(t-1)$$

$$\therefore y(2) = e^{-2\pi}\cos 2\sqrt{3}\pi + \frac{1}{\sqrt{3}\pi}e^{-\pi}\sin\sqrt{3}\pi \approx -0.00613$$

Question 6

a) *Consider the following system of differential equations,*

$$\frac{dx}{dt} = 2x + y, \qquad x(0) = 0,$$
$$\frac{dy}{dt} = x - 2y, \qquad y(0) = 1.$$

Find the solution for $x(t)$ *using the Laplace transform.*

b) *The weather of a typical day in Antarctica is classified into a normal day, a cold day and an extremely cold day. There is a 10% chance that a normal day turns into a cold day and a 10% chance that a normal day turns into an extremely cold day. The probability that a cold day remains a cold day is 70% and the probability that a cold turns into a normal day is x%. An extremely cold day has a 90% chance of staying extremely cold and it is impossible for an extremely cold day to turn normal.*
If today is extremely cold and the probability that 3 days from now is still extremely cold is 75.6%, find x.

a)

$$L\left(\frac{dx}{dt}\right) = sX - 0 = 2X + Y \quad \Rightarrow \quad (s-2)X = Y$$

$$L\left(\frac{dy}{dt}\right) = sY - 1 = X - 2Y \quad \Rightarrow \quad (s+2)Y = X + 1$$

$$(s+2)[(s-2)X] = X + 1 \quad \Rightarrow \quad (s^2 - 4)X - X = 1$$

$$X = \frac{1}{s^2 - 5} = \frac{1}{2\sqrt{5}}\left(\frac{1}{s-\sqrt{5}} - \frac{1}{s+\sqrt{5}}\right)$$

$$\therefore x(t) = L^{-1}(X) = \frac{\sqrt{5}}{10}\left(e^{\sqrt{5}t} - e^{-\sqrt{5}t}\right) = \frac{1}{\sqrt{5}}\sinh\sqrt{5}t$$

b)

$$M = \begin{pmatrix} 0.8 & x & 0 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.3 - x & 0.9 \end{pmatrix}, \qquad M^3 = \begin{pmatrix} & & \\ & & \\ & & 0.756 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 0.6 + 0.1x & 1.5x & 0.1x \\ 0.16 & 0.52 & 0.16 \\ 0.2 - 0.1x & 0.48 - 1.5x & 0.84 - 0.1x \end{pmatrix}\begin{pmatrix} 0.8 & x & 0 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.3 - x & 0.9 \end{pmatrix}$$

$$(0.48 - 1.5x)0.1 + (0.84 - 0.1x)0.9 = 0.756$$

$$\therefore x = 0.2 = 20\%$$

Question 7

a) *Given the matrix* $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ *has 3 distinct eigenvectors which form a basis in 3D, and given that 2 of its eigenvalues are 0 and* $\frac{1}{2}(3\sqrt{33}+15)$, *find the 3rd eigenvalues. Find also the eigenvector corresponding to the eigenvalues 0.*

b) *A bioengineer studies the interactions of 2 kinds of bacteria in a particular culture. Bacterium A feeds on bacterium B and depends on it for its food, while bacterium B depends only on sunlight. The bioengineer checks the numbers of A & B every hour; let* A_k *and* B_k *be these numbers, measured in millions, in the k-th hour. His model of the situation is given by the following equations:*

$$A_{k+1} = \frac{A_k}{2} + \frac{B_k}{100}, \qquad B_{k+1} = \frac{50A_k}{4} + \frac{5B_k}{4}$$

Initially, there are 50 million of type a and 5000 million of type B. By diagonalizing a matrix, compute how many bacteria of type A the model predicts there will be in 4 hours.

a) Trace, $1+5+9=15$

$$\lambda + 0 + \frac{1}{2}[3\sqrt{33}-15] = 15, \qquad \lambda = -\frac{1}{2}[3\sqrt{33}-15]$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ y \\ z \end{pmatrix} = 0 \quad \Rightarrow \quad y=-2, \qquad z=1$$

$$\therefore \textit{eigenvalue } -\frac{1}{2}[3\sqrt{33}-15] \textit{ and eigenvector } \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

b)

$$\begin{pmatrix} A_{k+1} \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{100} \\ -\frac{50}{4} & \frac{5}{4} \end{pmatrix}\begin{pmatrix} A_k \\ B_k \end{pmatrix}$$

$$\begin{pmatrix} A_4 \\ B_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{100} \\ -\frac{50}{4} & \frac{5}{4} \end{pmatrix}^4 \begin{pmatrix} 50 \\ 5000 \end{pmatrix}$$

$$\det\begin{pmatrix} \frac{1}{2}-\lambda & \frac{1}{100} \\ -\frac{50}{4} & \frac{5}{4}-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - \frac{7}{4}\lambda + \frac{3}{4} = 0 \quad \Rightarrow \quad \lambda = 1, \frac{3}{4}$$

$\lambda = 1,$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{100} \\ -\frac{50}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ y \end{pmatrix} = 0, \qquad y = 50$$

$\lambda = \frac{3}{4},$

$$\begin{pmatrix} -\frac{1}{4} & \frac{1}{100} \\ -\frac{50}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix} = 0, \qquad z = 25$$

$$P = \begin{pmatrix} 1 & 1 \\ 50 & 25 \end{pmatrix} \quad \Rightarrow \quad P^{-1} = \begin{pmatrix} -1 & \frac{1}{25} \\ 2 & -\frac{1}{25} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{100} \\ -\frac{50}{4} & \frac{5}{4} \end{pmatrix}^4 = \begin{pmatrix} 1 & 1 \\ 50 & 25 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{3}{4}\right)^4 \end{pmatrix}\begin{pmatrix} -1 & \frac{1}{25} \\ 2 & -\frac{1}{25} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 50 & 25 \end{pmatrix}\begin{pmatrix} -1 & \frac{1}{25} \\ \frac{81}{128} & -\frac{81}{6400} \end{pmatrix}$$

$$\therefore A_4 = 50\left(\frac{81}{128} - 1 + \frac{100}{25} - \frac{8100}{6400}\right) = 118.36\, million$$

Question 8

a) A chemical engineer has 2 tanks containing 100ℓ of water. Tank A initially contains water in which 25kg of a dangerous chemical are dissolved, and tank B contains x kg of this chemical. Pure water is poured into tank A at a constant rate of $4\,\ell\, min^{-1}$. The thoroughly mixed solution from tank A is constantly pumped into tank B at a rate of $6\,\ell\, min^{-1}$, while the solution from tank B is pumped back to tank A at a rate of $2\ell\, min^{-1}$. The solution in tank B is also pumped out and discarded at a rate of $4\ell\, min^{-1}$. The engineer wants to choose x in such a way that the ratio of the amount of the chemical in tank A to the amount in tank B is constant. Find x.

b) Classify the systems of linear ordinary equations with the following coefficient matrices:

$a)\ \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}\ b)\ \begin{pmatrix} 2 & -2 \\ 7 & 0 \end{pmatrix}\ c)\ \begin{pmatrix} -2 & -8 \\ 5 & 0 \end{pmatrix}\ d)\ \begin{pmatrix} -6 & 4 \\ -2 & 1 \end{pmatrix}\ e)\ \begin{pmatrix} 7 & -4 \\ 2 & -1 \end{pmatrix}$

a) Let x_A, x_B. be the number of kgs of the chemicals in tanks A, B.

$$\dot{x}_A = \frac{1}{100}(-6x_A + 2x_B), \qquad \dot{x}_B = \frac{1}{100}(6x_A - 6x_B)$$

$$\begin{pmatrix} \dot{x}_A \\ \dot{x}_B \end{pmatrix} = \frac{1}{100}\begin{pmatrix} -6 & 2 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \end{pmatrix}$$

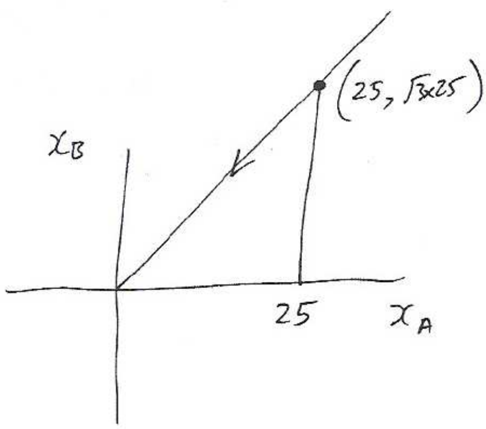
$$Tr\begin{pmatrix} -6 & 2 \\ 6 & -6 \end{pmatrix} = -\frac{12}{100}, \qquad \det\begin{pmatrix} -6 & 2 \\ 6 & -6 \end{pmatrix} = \frac{24}{100^2}$$

$$\left[Tr\begin{pmatrix} -6 & 2 \\ 6 & -6 \end{pmatrix}\right]^2 - 4\det\begin{pmatrix} -6 & 2 \\ 6 & -6 \end{pmatrix} = \frac{144}{100^2} - 4 \times \frac{24}{100^2} > 0$$

So it is a nodal sink.

The only way $\frac{x_A}{x_B}$ can be constant is if the trajectory in the phase plane is a straight line, which is only possible if that line is parallel to an eigenvector in the first quadrant.

$$\det\begin{pmatrix} -\frac{6}{100} - \lambda & \frac{2}{100} \\ \frac{6}{100} & -\frac{6}{100} - \lambda \end{pmatrix} = 0$$



$$100\lambda^2 + 12\lambda + 24 = 0 \quad \Rightarrow \quad \lambda = \frac{-6 \pm 2\sqrt{3}}{100}$$

Eigenvector,

$$\begin{pmatrix} \pm\frac{2\sqrt{3}}{100} & \frac{2}{100} \\ \frac{6}{100} & \pm\frac{2\sqrt{3}}{100} \end{pmatrix}\begin{pmatrix} 1 \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad y = \pm\sqrt{3}$$

Since we want the 1st quadrant, we are interested in the vector $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

The line has the equation $x_B = \sqrt{3}x_A$, so the amount of chemical initiall in the tank B should be $\sqrt{3} \times 25 \approx 43.30kg$.

b)

$a)\ \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}\ Trace = -1, det = -4 \quad \Rightarrow \quad Saddle$

$b)\ \begin{pmatrix} 2 & -2 \\ 7 & 0 \end{pmatrix}\ Trace = 2, det = 14, Tr^2 - 4det = 4 - 56 < 0 \quad \Rightarrow \quad Spiral\ Source$

$c)\ \begin{pmatrix} -2 & -8 \\ 5 & 0 \end{pmatrix}\ Trace = -2, det = 40, Tr^2 - 4det < 0 \quad \Rightarrow \quad Spiral\ Sink$

$d)\ \begin{pmatrix} -6 & 4 \\ -2 & 1 \end{pmatrix}\ Trace = -5, det = 2, Tr^2 - 4det > 0 \quad \Rightarrow \quad Nodal\ Sink$

$e)\ \begin{pmatrix} 7 & -4 \\ 2 & -1 \end{pmatrix}\ Trace = 6, det = 1, Tr^2 - 4det > 0 \quad \Rightarrow \quad Nodal\ Source$