Question 1

a) Let $y(x)$ be the solution of the initial value problem

$$2x - y\frac{dy}{dx} = 0, \qquad y \geq 0, \qquad y(0) = 2$$

Find the value of $y(4)$.

b) A hall of volume 1000cm^3 contains air with 0.001% of carbon monoxide. At time $t = 0$ the ventilation system starts blowing in air which contains 2% of carbon monoxide by volume. If the ventilation system blows in and extracts air at a rate of 0.3$cm^3 min^{-1}$, how long will it take for the air in the hall to contain 0.025% of carbon monoxide? Give your answer in minutes correct to 2 decimal places.

a) $y\,dy = 2x\,dx$

$$\frac{1}{2}y^2 = x^2 + c$$

$$y(0) = 2 \quad \Rightarrow \quad c = 2$$

$$y^2 = 2x^2 + 4$$

$$\therefore y(4) = \sqrt{2(4^2) + 4} = 6$$

b) Let $x\,m^3$ = volume of CO at time t.

$$\frac{dx}{dt} = 0.3 \cdot 0.02 - 0.3 \cdot \frac{x}{1000}$$

$$= 0.0003(20 - x)$$

$$\frac{dx}{20 - x} = 0.0003\,dt \quad \Rightarrow \quad -\ln|20 - x|$$

$$= 0.0003t + c$$

$$20 - x = Ae^{-0.0003t}$$

$$t = 0, \quad x = 0.01 \quad \Rightarrow \quad A = 19.99$$

$$x = 20 - 19.99e^{-0.0003t}$$

$$x = 0.25 \quad \Rightarrow \quad 19.75 = 19.99e^{-0.0003t}$$

$$\therefore t = \ln\left(\frac{19.75}{-0.0003(19.99)}\right) \approx 40.26\,mins$$

Question 2

a) *At 11am, a cup of coffee at 86˚ C is placed in an air-cond room which has a temperature of 23˚ C. The air-cond is immediately switched off and the room warms up at a uniform rate to 32˚ C at 12 noon. Assume that at any time between 11am and 12 noon, the rate of the coffee satisfies the equation:*

$$\frac{dT}{dt} = -(T - T_{env})$$

where T_{env} is the room temperature at that time and the unit of time is measured in hours, find the temperature of the coffee at 11.30am.

b) *Solve the differential equation $y'' - y = 2e^x$ with the initial conditions $y(0 = 2,\ y'(0) = 1$.*

a) T_{env} gets from 23° C to 32° C uniformly in 1 hour.

$T_{env} = 23 + 9t$

$\frac{dT}{dt} = -(T - 23 - 9t) \quad \Rightarrow \quad \frac{dT}{dt} + T = 23 + 9t$

$e^{\int dt} = e^t$

$T = e^{-t}\int e^t(23 + 9t)\,dt = e^{-t}\left(23e^t + \int 9te^t\,dt\right) = e^{-t}(23e^t + 9te^t - 9e^t + c)$

$T(0) = 86 \quad \Rightarrow \quad c = 72$

$T = 14 + 9t + 72e^{-t}$

$\therefore T\left(\frac{1}{2}\right) = 14 + \frac{9}{2} + 72e^{-0.5} \approx 62.17°C$

b) $\lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda = \pm 1$

$y = (Ax + B)e^x$

$y' = Ae^x + (Ax + B)e^x$

$y'' = 2Ae^x + (Ax + B)e^x$

$2Ae^x = e^x \quad \Rightarrow \quad A = 1$

$y = Ce^x + De^{-x} + xe^x$

$y' = Ce^x - De^{-x} + e^x + xe^x$

$y(0) = 2, y'(0) = 1 \quad \Rightarrow \quad C = D = 1$

$\therefore y = e^x + e^{-x} + xe^x$

Question 3

a) Consider the equation

$$\ddot{x} = \sec^2 x - 2, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Find the equation of the phase curve which satisfies the initial conditions $x(0) = 0,\ y(0) = 1$, *where* $y = \dot{x}$.

b) You have 20 000 bugs in a bottle. It is known that this bug population is given by a logistic model with a birth rate per capita of 25% per day. After a long time, you find that the population has attained an equilibrium of 100 000 bugs. Then you start an experiment with an antibiotic which kills 5 000 bugs a day. Find approximately how many bugs you will still have after another very long period of time. Give your answer correct to the nearest ten.

a) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right)$

$$\frac{d}{dx}\left(\frac{1}{2}y^2\right) = \sec^2 x - 2$$

$$\frac{1}{2}y^2 = \tan x - 2x + c$$

$$x = 0, y = 1 \quad \Rightarrow \quad c = \frac{1}{2}$$

$$\therefore\ y = \sqrt{2\tan x - 4x + 1}$$

b) $B = 0.25,\ \frac{B}{S} = 100000 \quad \Rightarrow \quad S = \frac{0.25}{100000}$

$$\frac{dN}{dt} = 0.25N - \frac{0.25}{100000}N^2 - 5000 = 0,$$

$$N^2 - 100000N + 2000000000 = 0, \qquad N = \frac{10^5 \pm 10^4\sqrt{20}}{2} \approx 27639.5{,}72360.5$$

$$\therefore\ N \approx 72360$$

Question 4

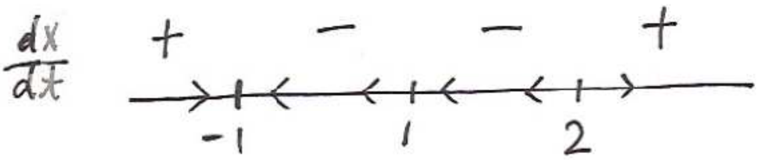
a) Find and classify all the equilibrium solutions of the differential equation

$$\frac{dx}{dt} = x^4 - 3x^3 + x^2 + 3x - 2.$$

b) Compute the inverse Laplace transform of

$$F(s) = \frac{s-2}{s^2 - 4s + 13}.$$

a) $\frac{dx}{dt} = (x-1)^2(x+1)(x-2)$



$x = -1$ *stable*, $\quad x = 1$ *unstable*, $\quad x = 2$ *unstable*.

b) $L^{-1}\left(\frac{s-2}{s^2-4s+13}\right) = L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) = e^{2t}\cos 3t$

Question 5

a) In an RLC circuit of inductance 1H, resistance 3Ω, capacitance $\frac{1}{2}$F and voltage V volts, it is known that the charge of the capacitor Q satisfies the equation

$$\frac{d^2Q}{dt^2} + 3\frac{dQ}{dt} + 2Q = V$$

At time $t = 0s$, you observe that there is no voltage applied to the circuit and that both $Q = \frac{dQ}{dt} = 0$ at that time. At time $t = 2s$, you apply a constant voltage of 2V to the circuit and then you switch off the voltage at time $t = 4s$. Find the value of Q at time $t = 5s$.

b) Morphine in the blood decomposes with a half life of 2.9 hours. Suppose a doctor injects 60mg of morphine into a patient and does it again 6 hours later. Find the amount of morphine in mg in the patient's blood at the seventh hour. You may assume that there is no morphine in the patient's blood prior to the first injection.

a) $\frac{d^2Q}{dt^2} + 3\frac{dQ}{dt} + 2Q = 2[u(t-2) - u(t-4)],\ Q(0) = \frac{dQ}{dt}(0) = 0$

$$L(Q)[s^2+3s+2]=\frac{2}{s}(e^{-2s}-e^{-4s})$$

$$L(Q)=\frac{2}{s(s+1)(s+2)}(e^{-2s}-e^{-4s})=\left(\frac{1}{s}-\frac{2}{s+1}+\frac{1}{s+2}\right)e^{-2s}-\left(\frac{1}{s}-\frac{2}{s+1}+\frac{1}{s+2}\right)e^{-4s}$$

$$Q=(1-2e^{-t+2}+e^{-2t+4})u(t-2)-(1-2e^{-t+4}+e^{-2t+8})u(t-4)$$

$$\therefore Q(5)=-2e^{-3}+e^{-6}+2e^{-1}-e^{-2}\approx 0.503$$

b) $\frac{dM}{dt}=-kM,$ half-life=2.9 hours

$$k=\frac{\ln 2}{2.9}$$

$$\frac{dM}{dt}=-kM+60\delta(t)+60\delta(t-6),\qquad M(0)=0$$

$$sL(M)=-kL(M)+60+60e^{-6s}$$

$$L(M)=\frac{60}{s+k}+\frac{60}{s+k}e^{-6s}$$

$$M=60e^{-kt}+60e^{-k(t-6)}u(t-6)$$

$$\therefore M(7)=60e^{-\frac{7\ln 2}{2.9}}+60e^{-\frac{\ln 2}{2.9}}\approx 58.5$$

Question 6

a) Where will the point $(1,2,3)$ *move to if we rotate 90˚ around the y-axis according to the right-hand rule and then rotate 90˚ around the z-axis according to the right-hand rule?*

b) A country has 2 main sectors in its economy: Agriculture (A) and Manufacturing (M). The demand for A and M are \$120 million per year and \$100 million per year respectively. It costs 40 cents of A to generate \$1 of A. It costs 30 cents of M to generate \$1 of M. It costs 30 cents of A to generate \$1 of M. It also costs 60 cents of M to generate \$1 of M. Formulate this as a Leontief input-output model and find the production in millions of dollars per year for A and M.

a) Rotate 90° around y-axis,

$$\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$$

Rotate 90° around z-axis,

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

b) $A = 0.4A + 0.3M + 120,\ M = 0.3A + 0.6M + 100$

$$\begin{pmatrix} A \\ M \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 \\ 0.3 & 0.6 \end{pmatrix}\begin{pmatrix} A \\ M \end{pmatrix} + \begin{pmatrix} 120 \\ 100 \end{pmatrix}$$

$$\begin{pmatrix} 0.6 & -0.3 \\ -0.3 & 0.4 \end{pmatrix}\begin{pmatrix} A \\ M \end{pmatrix} = \begin{pmatrix} 120 \\ 100 \end{pmatrix}$$

$$\therefore \begin{pmatrix} A \\ M \end{pmatrix} = \begin{pmatrix} 0.6 & -0.3 \\ -0.3 & 0.4 \end{pmatrix}^{-1}\begin{pmatrix} 120 \\ 100 \end{pmatrix} = \frac{1}{0.15}\begin{pmatrix} 0.4 & 0.3 \\ 0.3 & 0.6 \end{pmatrix}\begin{pmatrix} 120 \\ 100 \end{pmatrix} = \begin{pmatrix} 520 \\ 640 \end{pmatrix}$$

Question 7

a) Let a and b denote 2 constants. It is known that the matrix

$$\begin{pmatrix} 5 & 2 & a \\ \frac{32}{5} & 3 & -\frac{12}{5} \\ 10 & b & -2 \end{pmatrix}$$

has 3 distinct eigenvalues, that 1 of the eigenvalues is 2, and that its determinant is equal to -90. Find the other 2 eigenvalues.

b) A certain colony of rabbits is being attacked by foxes. Both rabbits and foxes are dying out due to a severe drought. Let R_n and F_n denote the number of rabbits and the number of foxes respectively in year n. It is known that these numbers satisfy the equations

$$R_{k+1} = \frac{1}{2}R_k - \frac{1}{4}F_k$$

$$F_{k+1} = \frac{1}{4}F_k$$

In the year $k = 0$, there are 8 192 rabbits and 1 024 foxes. When only 4 foxes survive, how many rabbits are there? Use linear algebra to solve this problem.

a) $2 + \lambda_1 + \lambda_2 = Tr = 6 \quad \Rightarrow \quad \lambda_1 + \lambda_2 = 4$

$2\lambda_1\lambda_2 = det = -90 \quad \Rightarrow \quad \lambda_1\lambda_2 = 45$

$\therefore \lambda_1 = -5, \qquad \lambda_2 = 9$

b) $\begin{pmatrix} R \\ F \end{pmatrix}_{k+1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} R \\ F \end{pmatrix}_k \quad \Rightarrow \quad \begin{pmatrix} R \\ F \end{pmatrix}_n = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix}^n \begin{pmatrix} R \\ F \end{pmatrix}_0$

$$\begin{vmatrix} \frac{1}{2} - \lambda & -\frac{1}{4} \\ 0 & \frac{1}{4} - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \left(\frac{1}{2} - \lambda\right)\left(\frac{1}{4} - \lambda\right) = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2}, \frac{1}{4}$$

$$\lambda = \frac{1}{2} \quad \Rightarrow \quad 0x - \frac{1}{4}y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = \frac{1}{4} \quad \Rightarrow \quad \frac{1}{4}x - \frac{1}{4}y = 0 \quad \Rightarrow \quad eigenvector \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2^n} & 0 \\ 0 & \frac{1}{4^n} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} R \\ F \end{pmatrix}_n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2^n} & 0 \\ 0 & \frac{1}{4^n} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 8192 \\ 1024 \end{pmatrix} = \begin{pmatrix} \frac{7168}{2^n} + \frac{1024}{4^n} \\ \frac{1024}{4^n} \end{pmatrix}$$

$$F_n = \frac{1024}{4^n} = 4 \quad \Rightarrow \quad n = 4$$

$$\therefore R_n = \frac{7168}{2^4} + \frac{1024}{4^4} = 452$$

Question 8

a) Classify the linear systems of ordinary differential equations with matrices $(i)\begin{pmatrix}1 & 3\\ 1 & -2\end{pmatrix}$, $(ii)\begin{pmatrix}2 & -2\\ 8 & 1\end{pmatrix}$, $(iii)\begin{pmatrix}2 & 2\\ 1 & 2\end{pmatrix}$, $(iv)\begin{pmatrix}0 & -2\\ 8 & -0.1\end{pmatrix}$, $(v)\begin{pmatrix}0 & -5\\ 5 & 0\end{pmatrix}$.

b) Humans and Klingons both live on Planet Zorg, but neither likes the other and both have a tendency to leave Zorg if they consider that the other side is too numerous. Despite this problem on Zorg, 1 million Humans and 2 million Klingons continue to move there from other plates as long as there are both Human and Klingon populations on Zorg. Suppose that we model this situation using the system of ordinary differential equations

$$\frac{dH}{dt} = 5H - 4K + 1, \qquad \frac{dK}{dt} = -H + 2K + 2$$

where H and K denote the number in millions of Humans and Klingons respectively at any time t and time is measured in years. If at time $t = 0$, *there are 500 million Humans and* $\frac{k}{6}$ *of Klingons living on Zorg, what is the minimum value that k must exceed in order that all Humans will be driven out of Zorg after some time?*

a) (i) $Tr = -1,\ det = -5 \ \Rightarrow\ \ Saddle$

(ii) $Tr = 3,\ det = 18,\ Tr^2 - 4det < 0 \ \Rightarrow\ \ Spiral\ Source$

(iii) $Tr = 4,\ det = 2,\ Tr^2 - 4det > 0 \ \Rightarrow\ \ Nodal\ Source$

(iv) $Tr = -0.1,\ det = 16,\ Tr^2 - 4det < 0 \ \Rightarrow\ \ Spiral\ Sink$

(v) $Tr = 0,\ det = 25,\ Tr^2 - 4det < 0 \ \Rightarrow\ \ Centre$

b) Let $\vec{r} = \begin{pmatrix}H\\ K\end{pmatrix}$, $B = \begin{pmatrix}5 & -4\\ -1 & 2\end{pmatrix}$, $\vec{F} = \begin{pmatrix}1\\ 2\end{pmatrix}$, then

$$\frac{d\vec{r}}{dt} = B\vec{r} + \vec{F} = B\vec{r} + BB^{-1}\vec{F} = B\left(\vec{r} + B^{-1}\vec{F}\right)$$

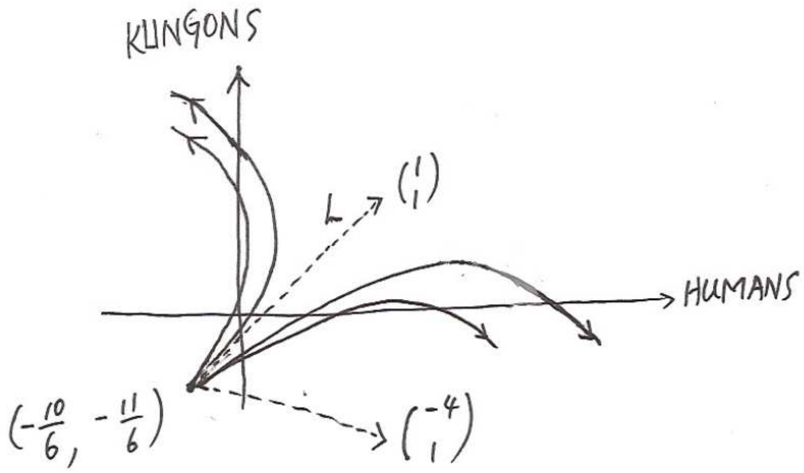
$Tr\ B = 7, \det B = 6, \qquad Tr^2 - 4det > 0 \ \Rightarrow\ \ Nodal\ Source$

$\begin{vmatrix}5-\lambda & -4\\ -1 & 2-\lambda\end{vmatrix} = 0, \qquad \lambda^2 - 7\lambda + 6 = 0 \ \Rightarrow\ \ \lambda = 1,6$

$\lambda = 1 \ \Rightarrow\ \ 4x - 4y = 0 \ \ \Rightarrow eigenvector \begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda = 6 \ \Rightarrow\ \ -x - 4y = 0 \ \ \Rightarrow eigenvector \begin{pmatrix}-4\\ 1\end{pmatrix}$

Translate axis to $-B^{-1}\vec{F} = -\frac{1}{6}\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{10}{6} \\ -\frac{11}{6} \end{pmatrix}$ as new origin



Equation of L is

$$K + \frac{11}{6} = H + \frac{10}{6}$$

To drive H to zero,

$$K + \frac{11}{6} > H + \frac{10}{6}$$

$$\frac{k}{6} > 50 + \frac{10}{6} - \frac{11}{6} = \frac{299}{6}$$

$$\therefore k > 299$$