First-order ordinary differential equations
First-degree first-order equations

First-degree first-order ODEs contain only $dy/dx$ equated to some function of $x$ and $y$, and can be written in either of two equivalent standard forms

$$\frac{dy}{dx} = F(x, y),$$

or

$$A(x, y) \, dx + B(x, y) \, dy = 0,$$

where $F(xmy) = -A(x, y)/B(x, y)$, and $F(x, y)$, $A(x, y)$ and $B(x, y)$ are in general functions of both $x$ and $y$. 
Separable-variable equations

A separable-variable equation is one which may be written in the conventional form

\[ \frac{dy}{dx} = f(x)g(y), \tag{1} \]

where \( f(x) \) and \( g(y) \) are functions of \( x \) and \( y \) respectively. Rearranging this equation, we obtain

\[ \int \frac{dy}{g(y)} = \int f(x) \, dx. \]

Finding the solution \( y(x) \) that satisfies Eq. (1) then depends only on the ease with which the integrals in the above equation can be evaluated.
Example

Solve

\[ \frac{dy}{dx} = x + xy. \]

Answer

Since the RHS of this equation can be factorized to give \( x(1 + y) \), the equation becomes separable and we obtain

\[ \int \frac{dy}{1 + y} = \int x \, dx \]

Now integrating both sides, we find

\[ \ln(1 + y) = \frac{x^2}{2} + c, \]

and so

\[ 1 + y = \exp \left( \frac{x^2}{2} + c \right) = A \exp \left( \frac{x^2}{2} \right), \]

where \( c \) and hence \( A \) is an arbitrary constant.
 Exact equation

An exact first-degree first-order ODE is one of the form

\[ A(x, y) \, dx + B(x, y) \, dy = 0 \quad \text{and for which} \quad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{2} \]

In this case, \( A(x, y) \, dx + B(x, y) \, dy \) is an exact differential, \( dU(x, y) \) say. That is,

\[ A \, dx + B \, dy = dU = \frac{\partial U}{\partial x} \, dx + \frac{\partial U}{\partial y} \, dy, \]

from which we obtain

\[ A(x, y) = \frac{\partial U}{\partial x}, \tag{3} \]

\[ B(x, y) = \frac{\partial U}{\partial y}. \tag{4} \]
Since $\partial^2 U/\partial x \partial y = \partial^2 U/\partial y \partial x$, we therefore require

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \quad (5)$$

If Eq. (5) holds then Eq. (2) can be written

$$dU(x, y) = 0,$$

which has the solution $U(x, y) = c$, where $c$ is a constant and from Eq. (3), $U(x, y)$ is given by

$$U(x, y) = \int A(x, y) \, dx + F(y). \quad (6)$$

The function $F(y)$ can be found from Eq. (4) by differentiating Eq. (6) with respect to $y$ and equating to $B(x, y)$. 
Example

Solve

\[ x \frac{dy}{dx} + 3x + y = 0. \]

Answer

Rearranging into the form Eq. (2), we have

\[(3x + y) \, dx + x \, dy = 0, \]

i.e. \( A(x, y) = 3x + y \) and \( B(x, y) = x \). Since \( \frac{\partial A}{\partial y} = 1 = \frac{\partial B}{\partial x} \), the equation is exact, and by Eq. (6), the solution is given by

\[ U(x, y) = \int (3x + y) \, dx + F(y) = c_1 \]

\[ \Rightarrow \frac{3x^2}{2} + xy + F(y) = c_1. \]
Differentiating \( U(x, y) \) with respect to \( y \) and equating it to \( B(x, y) = x \), we obtain \( dF/dy = 0 \), which integrates to give \( F(y) = c_2 \). Therefore, letting \( c = c_1 - c_2 \), the solution to the original ODE is

\[
\frac{3x^2}{2} + xy = c.
\]
Inexact equations: integrating factors

Equations that may be written in the form

\[ A(x, y) \, dx + B(x, y) \, dy = 0 \]  
but for which \( \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x} \) \hspace{1cm} (7)

are known as inexact equations. However the differential \( A \, dx + B \, dy \) can always be made exact by multiplying by an integrating factor \( \mu(x, y) \) that obeys

\[ \frac{\partial (\mu A)}{\partial y} = \frac{\partial (\mu B)}{\partial x} . \]  \hspace{1cm} (8)

For an integrating factor that is a function of both \( x \) and \( y \), there exists no general method for finding it. If, however, an integrating factor exists that is a function of either \( x \) or \( y \) alone, then Eq. (8) can be solved to find it.
For example, if we assume that the integrating factor is a function of $x$ alone, $\mu = \mu(x)$, then from Eq. (8),

$$\mu \frac{\partial A}{\partial y} = \mu \frac{\partial B}{\partial x} + B \frac{d\mu}{dx}.$$ 

Rearranging this expression we find

$$\frac{d\mu}{\mu} = \frac{1}{B} \left( \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} \right) dx = f(x) \, dx,$$

where we require $f(x)$ also to be a function of $x$ only. The integrating factor is then given by

$$\mu(x) = \exp \left\{ \int f(x) \, dx \right\} \text{ where } f(x) = \frac{1}{B} \left( \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} \right).$$ \hspace{1cm} (9)

Similarly, if $\mu = \mu(y)$, then

$$\mu(y) = \exp \left\{ \int g(y) \, dy \right\} \text{ where } g(y) = \frac{1}{A} \left( \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right).$$ \hspace{1cm} (10)
Example

Solve

\[ \frac{dy}{dx} = -\frac{2}{y} - \frac{3y^2}{2x}. \]

Answer

Rearranging into the form Eq. (7), we have

\[(4x + 3y^2)\, dx + 2xy\, dy = 0, \]  
\[(11)\]
i.e. \( A(x, y) = 4x + 3y^2 \) and \( B(x, y) = 2xy. \)

Therefore,

\[ \frac{\partial A}{\partial y} = 6y, \quad \frac{\partial B}{\partial x} = 2y, \]
so the ODE is not exact in its present form.

However, we see that

\[ \frac{1}{B} \left( \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} \right) = \frac{2}{x}, \]

a function of \( x \) alone.
Therefore an integrating factor exists that is also a function of \( x \) alone and, ignoring the arbitrary constant, is given by

\[
\mu(x) = \exp \left\{ 2 \int \frac{dx}{x} \right\} = \exp(2 \ln x) = x^2.
\]

Multiplying Eq. (11) through by \( \mu(x) = x^2 \), we obtain

\[
(4x^3 + 3x^2y^2) \, dx + 2x^3y \, dy = 0.
\]

By inspection, this integrates to give the solution

\[
x^4 + y^2x^3 = c,
\]

where \( c \) is a constant.
Linear equations

Linear first-order ODEs are a special case of inexact ODEs and can be written in the conventional form

\[ \frac{dy}{dx} + P(x)y = Q(x). \] (12)

Such equations can be made exact by multiplying through by an appropriate integrating factor which is always a function of \( x \) alone. An integrating factor \( \mu(x) \) must be such that

\[ \mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x), \] (13)

which may then be integrated directly to give

\[ \mu(x)y = \int \mu(x)Q(x) \, dx. \] (14)
The required integrating factor $\mu(x)$ is determined by the first equality in Eq. (13),

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P y,$$

which gives the simple relation

$$\frac{d\mu}{dx} = \mu(x)P(x) \Rightarrow \mu(x) = \exp \left\{ \int P(x) \, dx \right\}.$$  

(15)
Example

Solve

\[ \frac{dy}{dx} + 2xy = 4x. \]

Answer

The integrating factor is given by

\[ \mu(x) = \exp\left\{ \int 2x \, dx \right\} = \exp x^2. \]

Multiplying through the ODE by \( \mu(x) = \exp x^2 \), and integrating, we have

\[ y \exp x^2 = 4 \int x \exp x^2 \, dx = 2 \exp x^2 + c. \]

The solution to the ODE is therefore given by \( y = 2 + c \exp(-x^2) \).
Homogeneous equations

Homogeneous equations are ODEs that may be written in the form

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)} = F \left( \frac{y}{x} \right),$$

(16)

where $A(x, y)$ and $B(x, y)$ are homogeneous functions of the same degree. A function $f(x, y)$ is homogeneous of degree $n$ if, for any $\lambda$, it obeys

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

For example, if $A = x^2y - xy^2$ and $B = x^3 + y^3$ then we see that $A$ and $B$ are both homogeneous functions of degree 3.
The RHS of a homogeneous ODE can be written as a function of $y/x$. The equation can then be solved by making the substitution $y = vx$ so that

$$\frac{dy}{dx} = v + x \frac{dv}{dx} = F(v).$$

This is now a separable equation and can be integrated to give

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}. \quad (17)$$
Example

Solve
\[
\frac{dy}{dx} = \frac{y}{x} + \tan \left( \frac{y}{x} \right).
\]

Answer

Substituting \( y = vx \), we obtain

\[
v + x \frac{dv}{dx} = v + \tan v.
\]

Cancelling \( v \) on both sides, rearranging and integrating gives

\[
\int \cot v \, dv = \int \frac{dx}{x} = \ln x + c_1.
\]

But

\[
\int \cot v \, dv = \int \frac{\cos v}{\sin v} \, dv = \ln(\sin v) + c_2,
\]

so the solution to the ODE is \( y = x \sin^{-1} Ax \), where \( A \) is a constant.
Isobaric equations

An isobaric ODE is a generalization of the homogeneous ODE and is of the form

\[ \frac{dy}{dx} = \frac{A(x, y)}{B(x, y)}, \]  

(18)

where the RHS is dimensionally consistent if \( y \) and \( dy \) are each given a weight \( m \) relative to \( x \) and \( dx \), i.e. if the substitution \( y = vx^m \) makes the equation separable.
Example

Solve

\[ \frac{dy}{dx} = -\frac{1}{2yx} \left( y^2 + \frac{2}{x} \right). \]

Answer

Rearranging we have

\[ \left( y^2 + \frac{2}{x} \right) dx + 2yx dy = 0, \]

Giving \( y \) and \( dy \) the weight \( m \) and \( x \) and \( dx \) the weight 1, the sums of the powers in each term on the LHS are \( 2m + 1 \), \( 0 \) and \( 2m + 1 \) respectively. These are equal if \( 2m + 1 = 0 \), i.e. if \( m = -\frac{1}{2} \).

Substituting \( y = vx^m = vx^{-1/2} \), with the result that \( dy = x^{-1/2} dv - \frac{1}{2}vx^{-3/2} dx \), we obtain

\[ v \, dv + \frac{dx}{x} = 0, \]

which is separable and integrated to give

\[ \frac{1}{2}v^2 + \ln x = c. \]

Replacing \( v \) by \( y\sqrt{x} \), we obtain the solution

\[ \frac{1}{2}y^2x + \ln x = c. \]
Bernoulli’s equation

Bernoulli’s equation has the form

\[ \frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{where } n \neq 0 \text{ or } 1 \]  (19)

This equation is non-linear but can be made linear by substitution \( v = y^{1-n} \), so that

\[ \frac{dy}{dx} = \left( \frac{y^n}{1-n} \right) \frac{dv}{dx}. \]

Substituting this into Eq. (19) and dividing through by \( y^n \), we find

\[ \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x), \]

which is a linear equation, and may be solved.
Example

Solve
\[ \frac{dy}{dx} + \frac{y}{x} = 2x^3 y^4. \]

Answer

If we let \( v = y^{1-4} = y^{-3} \), then
\[ \frac{dy}{dx} = -\frac{y^4}{3} \frac{dv}{dx}. \]

Substituting this into the ODE and rearranging, we obtain
\[ \frac{dv}{dx} - \frac{3v}{x} = -6x^3. \]

Multiplying through by the following integrating factor
\[ \exp \left\{ -3 \int \frac{dx}{x} \right\} = \exp(-3 \ln x) = \frac{1}{x^3}, \]
the solution is then given by
\[ \frac{v}{x^3} = -6x + c. \]

Since \( v = y^{-3} \), we obtain \( y^{-3} = -6x^4 + cx^3. \)
Miscellaneous equations

\[ \frac{dy}{dx} = F(ax + by + c), \quad (20) \]

where \(a, b\) and \(c\) are constants, i.e. \(x\) and \(y\) appear on the RHS in the particular combination \(ax + by + c\) and not in any other combination or by themselves. This equation can be solved by making the substitution \(v = ax + by + c\), in which case

\[ \frac{dv}{dx} = a + b \frac{dy}{dx} = a + bF(v), \quad (21) \]

which is separable and may be integrated directly.
Example

Solve

\[ \frac{dy}{dx} = (x + y + 1)^2. \]

Answer

Making the substitution \( v = x + y + 1 \), from Eq. (21), we obtain

\[ \frac{dv}{dx} = v^2 + 1, \]

which is separable and integrates to give

\[ \int \frac{dv}{1 + v^2} = \int dx \implies \tan^{-1} v = x + c_1. \]

So the solution to the original ODE is

\[ \tan^{-1}(x + y + 1) = x + c_1, \]

where \( c_1 \) is a constant of integration.
Miscellaneous equations (continued)

We now consider

\[
\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g},
\]  

(22)

where \(a, b, c, e, f\) and \(g\) are all constants. This equation may be solved by letting \(x = X + \alpha\) and \(y = Y + \beta\), where \(\alpha\) and \(\beta\) are constants found from

\[
a\alpha + b\beta + c = 0 \tag{23}
\]

\[
e\alpha + f\beta + g = 0. \tag{24}
\]

Then Eq. (22) can be written as

\[
\frac{dY}{dX} = \frac{aX + bY}{eX + fY'},
\]

which is homogeneous and may be solved.
Example

Solve

\[
\frac{dy}{dx} = \frac{2x - 5y + 3}{2x + 4y - 6}.
\]

Answer

Let \( x = X + \alpha \) and \( y = Y + \beta \), where \( \alpha \) and \( \beta \) obey the relations

\[
2\alpha - 5\beta + 3 = 0
\]

\[
2\alpha + 4\beta - 6 = 0,
\]

which solve to give \( \alpha = \beta = 1 \). Making these substitutions we find

\[
\frac{dY}{dX} = \frac{2X - 5Y}{2X + 4Y},
\]

which is a homogeneous ODE and can be solved by substituting \( Y = vX \) to obtain

\[
\frac{dv}{dX} = \frac{2 - 7v - 4v^2}{X(2 + 4v)}.
\]
This equation is separable, and using partial fractions, we find

\[
\int \frac{2 + 4v}{2 - 7v - 4v^2} \, dv = -\frac{4}{3} \int \frac{dv}{4v - 1} - \frac{2}{3} \int \frac{dv}{v + 2}
\]

\[= \int \frac{dX}{X},\]

which integrates to give

\[
\ln X + \frac{1}{3} \ln(4v - 1) + \frac{2}{3} \ln(v + 2) = c_1,
\]

or

\[
X^3(4v - 1)(v + 2)^2 = 3c_1.
\]

Since \( Y = vX, x = X + 1 \) and \( y = Y + 1 \), the solution to the original ODE is given by

\[
(4y - x - 3)(y + 2x - 3)^2 = c_2, \text{ where } c_2 = 3c_1.
\]
Higher-degree first-order equations

Higher-degree first-order equations can be written as $F(x, y, dy/dx) = 0$. The most general standard form is

$$p^n + a_{n-1}(x, y)p^{n-2} + \cdots + a_1(x, y)p + a_0(x, y) = 0,$$

(25)

where $p = dy/dx$. 

Equations soluble for $p$

Sometime the LHS of Eq. (25) can be factorized into

$$(p - F_1)(p - F_2) \cdots (p - F_n) = 0,$$  \hspace{1cm} (26)

where $F_i = F_i(x, y)$. We are then left with solving the $n$ first-degree equations $p = F_i(x, y)$. Writing the solutions to these first-degree equations as $G_i(x, y) = 0$, the general solution to Eq. (26) is given by the product

$$G_1(x, y)G_2(x, y) \cdots G_n(x, y) = 0.$$  \hspace{1cm} (27)
Example

Solve

\((x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp + 2xy^2 = 0.\)  \hspace{1cm} (28)

Answer

This equation may be factorized to give

\[ \left[(x + 1)p - y\right]\left[(x^2 + 1)p - 2xy\right] = 0. \]

Taking each bracket in turn we have

\[(x + 1)\frac{dy}{dx} - y = 0,\]

\[(x^2 + 1)\frac{dy}{dx} - 2xy = 0,\]

which have the solutions \(y - c(x + 1) = 0\) and \(y - c(x^2 + 1) = 0\) respectively. The general solution to Eq. (28) is then given by

\[ [y - c(x + 1)][y - c(x^2 + 1)] = 0. \]
Equations soluble for $x$

Equations that can be solved for $x$, i.e. such that they may be written in the form

$$x = F(y, p),$$

(29)

can be reduced to first-degree equations in $p$ by differentiating both sides with respect to $y$, so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p} \frac{dp}{dy}.$$

This results in an equation of the form $G(y, p) = 0$, which can be used together with Eq. (29) to eliminate $p$ and give the general solution.
Example

Solve

\[ 6y^2 p^2 + 3xp - y = 0. \]  \hfill (30)

Answer

This equation can be solved for \( x \) explicitly to give \( 3x = y/p - 6y^2p \). Differentiating both sides with respect to \( y \), we find

\[ 3 \frac{dx}{dy} = \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2} \frac{dp}{dy} - 6y^2 \frac{dp}{dy} - 12yp, \]

which factorizes to give

\[ (1 + 6yp^2) \left( 2p + y \frac{dp}{dy} \right) = 0. \]  \hfill (31)

Setting the factor containing \( dp/dy \) equal to zero gives a first-degree first-order equation in \( p \), which may be solved to give \( py^2 = c \). Substituting for \( p \) in Eq. (30) then yields the general solution of Eq. (30):

\[ y^3 = 3cx + 6c^2. \]  \hfill (32)
If we now consider the first factor in Eq. (31), we find $6p^2 y = -1$ as a possible solution. Substituting for $p$ in Eq. (30) we find the singular solution

$$8y^3 + 3x^2 = 0.$$ 

Note that the singular solution contains no arbitrary constants and cannot be found from the general solution (32) by any choice of the constant $c$. 
Equations soluble for $y$

Equations that can be solved for $y$, i.e. such that they may be written in the form

$$y = F(x, p), \quad (33)$$

can be reduced to first-degree first-order equations in $p$ by differentiating both sides with respect to $y$, so that

$$\frac{dy}{dx} = p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} \frac{dp}{dx}.$$ 

This results in an equation of the form $G(x, y) = 0$, which can be used together with Eq. (33) to eliminate $p$ and give the general solution.
Example

Solve

\[ xp^2 + 2xp - y = 0. \] (34)

Answer

This equation can be solved for \( y \) explicitly to give \( y = xp^2 + 2xp \). Differentiating both sides with respect to \( x \), we find

\[
\frac{dy}{dx} = p = 2xp \frac{dp}{dx} + p^2 + 2x \frac{dp}{dx} + 2p,
\]

which after factorizing gives

\[
(p + 1) \left( p + 2x \frac{dp}{dx} \right) = 0. \] (35)
To obtain the general solution of Eq. (34), we first consider the factor containing \( \frac{dp}{dx} \). This first-degree first-order equation in \( p \) has the solution \( xp^2 = c \), which we then use to eliminate \( p \) from Eq. (34). We therefore find that the general solution to Eq. (34) is

\[
(y - c)^2 = 4cx. \tag{36}
\]

If we now consider the first factor in Eq. (35), we find this has the simple solution \( p = -1 \). Substituting this into Eq. (34) then gives

\[
x + y = 0,
\]

which is a singular solution to Eq. (34).
Clairaut’s equation

The Clairaut’s equation has the form

$$y = px + F(p),$$  \hspace{1cm} (37)

and is therefore a special case of equations soluble for $y$, Eq. (33).

Differentiating Eq. (37) with respect to $x$, we find

$$\frac{dy}{dx} = p = p + x \frac{dp}{dx} + \frac{dF}{dp} \frac{dp}{dx},$$

$$\Rightarrow \frac{dp}{dx} \left( \frac{dF}{dp} + x \right) = 0. \hspace{1cm} (38)$$

Considering first the factor containing $dp/dx$, we find

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \Rightarrow y = c_1x + c_2. \hspace{1cm} (39)$$

Since $p = dy/dx = c_1$, if we substitute Eq. (39) into Eq. (37), we find $c_1x + c_2 = c_1x + F(c_1)$. 


Therefore the constant $c_2$ is given by $F(c_1)$, and the general solution to Eq. (37)

$$y = c_1 x + F(c_1), \quad (40)$$

e.i. the general solution to Clairaut’s equation can be obtained by replacing $p$ in the ODE by the arbitrary constant $c_1$. Now considering the second factor in Eq. (38), also have

$$\frac{dF}{dp} + x = 0, \quad (41)$$

which has the form $G(x, p) = 0$. This relation may be used to eliminate $p$ from Eq. (37) to give a singular solution.
Example

Solve

\[ y = px + p^2. \]  \hfill (42)

Answer

From Eq. (40), the general solution is \( y = cx + c^2 \).
But from Eq. (41), we also have
\[ 2p + x = 0 \Rightarrow p = -x/2. \] Substituting this into
Eq. (42) we find the singular solution \( x^2 + 4y = 0 \).