PC1134 Lecture 22

**Topic:**

Green’s theorem in the plane

**Applications:**

Relation between a double integral and line integral along the boundary of the area.

Evaluate either a line integral around a closed path or a double integral over the area enclosed, whichever is easier to do.

**Scope**

- Green’s Theorem in a plane
- Proof
- Examples
Green’s Theorem

If

1. the region of integration is *simple*; and

2. \( P = P(x, y) \) and \( Q = Q(x, y) \) have continuous first partial derivatives at every point of this region.

then

\[
\iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint P \, dx + Q \, dy
\]

\( \oint \) means integration around a closed curve back to the starting point.

The line integral is in the counter-clockwise direction.
Proof

\[ \iint \frac{\partial P}{\partial y} \, dx \, dy = \int_{a}^{b} dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} \, dy \]

\[ = \int_{a}^{b} [P(x, y_2) - P(x, y_1)] \, dx \]

\[ \oint P \, dx = \int_{AB} P \, dx + \int_{BA} P \, dx \]

\[ = \int_{a}^{b} P[x, y_1(x)] \, dx + \int_{a}^{b} P[x, y_2(x)] \, dx \]

\[ = - \int_{a}^{b} P[x, y_2(x)] \, dx - \int_{b}^{a} P[x, y_1(x)] \, dx \]

\[ \iint \frac{\partial P}{\partial y} \, dx \, dy = - \oint P \, dx \]

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Proof (cont.)

\[ \iint \frac{\partial P}{\partial y} \, dxdy = - \int P \, dx \]

Similarly it can be shown that

\[ \iint \frac{\partial Q}{\partial x} \, dxdy = \oint Q \, dy \]

\[ \Downarrow \]

\[ \iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy = \oint P \, dx + Q \, dy \]
Example

Area of an ellipse

\[ x = a \cos \theta \]
\[ y = b \sin \theta \]

\[ \iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint P \, dx + Q \, dy \]

Choose \( Q = x, \) \( P = -y, \)

\[ \iint dx \, dy = \frac{1}{2} \oint x \, dy - y \, dx \]
\[ = \frac{1}{2} \oint a \cos \theta \, b \cos \theta \, d\theta + b \sin \theta \, a \cos \theta \, d\theta \]
\[ = \frac{1}{2} \int_{0}^{2\pi} ab(\cos^2 \theta + \sin^2 \theta) \, d\theta \]
\[ = \frac{1}{2} ab \int_{0}^{2\pi} d\theta = \pi ab \]
Example

Let

\[ P = F_x, \quad Q = F_y \]

\[ \oint P \, dx + Q \, dy = \iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy \]

\[ \oint (F_x \, dx + F_y \, dy) = \iint \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \, dxdy \]

\[ \oint \vec{F} \cdot d\vec{r} = \iint \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \, dxdy \]

If

\[ \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} \]

then

\[ \oint \vec{F} \cdot d\vec{r} = 0 \quad \implies \quad \text{conservative force} \]

\[ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \] is the \( z \)-component of \( \nabla \times \vec{F} \). In 2D,

\[ \nabla \times \vec{F} = \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} \]