PC1134 Lecture 17

Topic

Line integrals

Application example

Work done by a force.

\[ dW = \vec{F} \cdot d\vec{r} \]

\[ W = \int_{A}^{B} \vec{F} \cdot d\vec{r} \]

Scope

- Review of integration
- Line integral
- Calculation of line integral
Integration

The definite integral of \( f(x) \) between the lower limit \( x = a \) and the upper limit \( x = b \). \( f(x) \) is called the integrand.

\[
I = \int_{a}^{b} f(x) \, dx
\]

\( I \) is the area under the curve.
Integration from First Principles

Divide the area into slices of width $dx$ (not required to be uniform). The area of each slide is

$$dA \approx f(x)dx$$

Total area

$$A \approx \sum f(x)\Delta x$$

The error is reduced when $\Delta x$ is small. In the limit of $\Delta x \to 0$,

$$A = \lim_{\Delta x \to 0} \sum f(x)dx = \int_{a}^{b} f(x)dx$$
Properties of Integration

\[ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \]

\[ \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \]

\[ \int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \]

\[ \frac{d}{dx} \left[ \int_a^x f(t)\,dt \right] = f(x) \]
Integration of Simple Functions

\[
\int adx = ax \\
\int x^n \, dx = \frac{1}{n + 1} x^{n+1} \\
\int e^{ax} \, dx = \frac{e^{ax}}{a} \\
\int \frac{dx}{x} = \ln x \\
\int \cos ax \, dx = \frac{\sin ax}{a} \\
\int \sin ax \, dx = -\frac{\cos ax}{a} \\
\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \\
\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right)
\]

......
Integration by Substitution

Example 1

\[ I = \int \frac{dx}{\sqrt{1 - x^2}} \]

Let

\[ x = \sin u \]

then

\[ dx = \cos u \, du \]

\[ I = \int \frac{\cos u \, du}{\sqrt{1 - \sin^2 u}} = \int \frac{\cos u \, du}{\cos^2 u} = \int du = u \]

\[ I = \sin^{-1} x \]

Example 2

\[ I = \int \frac{dx}{x^2 + 4x + 7} = \int \frac{dx}{(x + 2)^2 + 3} \]

Let \( y = x + 2 \), then \( dy = dx \) and

\[ I = \int \frac{dy}{y^2 + 3} = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{y}{\sqrt{3}} \right) = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x + 2}{\sqrt{3}} \right) \]
Integration by Parts

If $u$ and $v$ are functions of $x$

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx}v$$

Integrate both sides

$$uv = \int u \frac{dv}{dx} dx + \int \frac{du}{dx}v dx$$

This can be rewritten as

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx}v dx$$

Example

$$I = \int x \sin x dx = \int x \frac{d(-\cos x)}{dx} dx = x(-\cos x)$$

$$- \int (1)(-\cos x) dx = -x \cos x + \sin x$$
Work Done by a Varying Force

\[ dW = \vec{F} \cdot d\vec{r} \]

\[ W = \int \vec{F} \cdot d\vec{r} \]

- \( \vec{F} \) is a function of position
  
  \[ \vec{F}(\vec{r}) \text{ or } \vec{F}(x, y, z) \]

- Object is restricted to move along the curved path.
Line Integral

\[ W = \int \vec{F} \cdot d\vec{r} \]

2D

A curve is described by one equation

\[ \phi(x, y) = 0 \quad \implies \quad y = \psi(x) \]

\[ \int \vec{F}(x, y) \cdot d\vec{r} \quad \implies \quad \int f(x)dx \]

or

\[ \int \vec{F}(x, y) \cdot d\vec{r} \quad \implies \quad \int f(s)ds \]

3D

Two equations relating \( x, y \) and \( z \) are required to describe a curve

\[ \implies \text{Only one independent variable} \]

\[ \implies \text{One dimensional integral} \]
Example

\[ \vec{F} = xy\hat{x} - y^2\hat{y} \]

Calculate

\[ W = \int \vec{F} \cdot d\vec{r} \]

along

1. Path 1 (black) \( y = x/2 \)
2. Path 2 (red) \( y = x^2/4 \)
3. Path 3 (green) \( x = 0 \) and \( y = 1 \)
4. Path 4 (blue) \( x = 2t^3, \ y = t^2 \).
Example

Path 1 (black)

\[ y = \frac{1}{2} x, \quad dy = \frac{1}{2} dx \]

\[ d\vec{r} = dx\hat{x} + dy\hat{y} = dx\hat{x} + \frac{1}{2} dx\hat{y} = \left( \hat{x} + \frac{1}{2} \hat{y} \right) dx \]

\[ \vec{F} \cdot d\vec{r} = F_x dx + F_y dy = xy dx - y^2 \frac{1}{2} dx = \left( xy - \frac{1}{2} y^2 \right) dx \]

\[ = \left( \frac{1}{2} x^2 - \frac{1}{8} x^2 \right) dx = \frac{3}{8} x^2 dx \]

\[ W_1 = \int \vec{F} \cdot d\vec{r} = \int_0^2 \frac{3}{8} x^2 dx = \frac{x^3}{8} \bigg|_0^2 = 1 \]
Example

Path 2 (red)

\[
y = \frac{1}{4} x^2, \quad dy = \frac{1}{2} x \, dx
\]

\[
d\vec{r} = dx \, \hat{x} + dy \, \hat{y} = dx \, \hat{x} + \frac{1}{2} x dx \, \hat{y} = \left( \hat{x} + \frac{1}{2} x \hat{y} \right) \, dx
\]

\[
\vec{F} \cdot d\vec{r} = F_x dx + F_y dy
\]

\[
= \left( xy - \frac{1}{2} xy^2 \right) \, dx = \left( \frac{1}{4} x^3 - \frac{1}{32} x^5 \right) \, dx
\]

\[
W_2 = \int \vec{F} \cdot d\vec{r} = \int_0^2 \left( \frac{1}{4} x^3 - \frac{1}{32} x^5 \right) \, dx = \frac{2}{3}
\]
Example

Path 3 (green)

Along the vertical segment, \( x = 0 \) and \( dx = 0 \)

\[
\vec{F} \cdot d\vec{r} = F_x dx + F_y dy = -y^2 dy
\]

\[
W_{3a} = \int \vec{F} \cdot d\vec{r} = \int_0^1 (-y^2) dy = -\frac{1}{3}
\]

Along the horizontal segment, \( y = 1 \) and \( dy = 0 \)

\[
\vec{F} \cdot d\vec{r} = F_x dx + F_y dy = x dx
\]

\[
W_{3b} = \int \vec{F} \cdot d\vec{r} = \int_0^2 x dx = 2
\]

\[
W_3 = W_{3a} + W_{3b} = -\frac{1}{3} + 2 = \frac{5}{3}
\]
Example

Path 4 (blue):

\[ x = 2t^3, \quad y = t^2 \]

\[ \vec{F} = xy\hat{x} - y^2\hat{y} = 2t^5\hat{x} - t^4\hat{y} \]

\[ d\vec{r} = dx\hat{x} + dy\hat{y} = 6t^2\,dt\hat{x} + 2tdt\hat{y} \]

\[ \vec{F} \cdot d\vec{r} = F_x dx + F_y dy = 12t^7\,dt - 2t^5\,dt \]

\[ W_4 = \int \vec{F} \cdot d\vec{r} = 12 \int_0^1 t^7\,dt - 2 \int t^5\,dt = \frac{7}{6} \]
Summary

• $\vec{F}(x, y, z)$ is a function of one variable because $x$, $y$ and $z$ are related.

• This variable can be one of $x$, $y$ and $z$ or can be something else.

• The integration over $\vec{F} \cdot d\vec{r}$ is an one-dimensional integral over this variable.

• The line integral between two points can be path dependent. ($W_1 = 1$, $W_2 = 2/3$, $W_3 = 5/3$, $W_4 = 7/6$).