PC1134 Lecture 15

Topic

Field, Potential & Gradient

Objectives

To understand the concept of gradient, its geometrical and physical meanings; and to be able to apply it to force/field and potential.

Relevance

- Field and potential
- Gradient, rate of change of a function along any direction
- Differential relationship between force/field and potential
Gravitational Force  
(near the surface of Earth)

\[ \vec{F} = -mg\hat{z} \]

Gravitational field:

\[ \vec{g} = -g\hat{z} \]

Potential:

\[ gz = U(z) \]

Potential energy:

\[ mgz \]

Equipotential surface:

\[ U(z) = gz = \text{const} \]

\[ \implies z = \text{const.} \]
Field & Potential of a Point Charge

Electrostatic force between two point charges:

$$\vec{F} = \frac{kq_1q_2}{r^2}\hat{r}$$

Electric field:

$$\vec{E} = \frac{kq}{r^2}\hat{r}$$

Potential:

$$V(r) = \frac{kq}{r}$$

Equipotential surface:

$$V(r) = \frac{kq}{r} = \text{const.}$$

$$\implies r = \text{const.}$$
Potential of Two Charges

How does the potential change along any direction?
Potential

Potential at $P_0$: $\phi(x_0, y_0, z_0)$
Potential at $P$: $\phi(x, y, z)$

Let

$$\vec{r} - \vec{r}_0 = s \hat{u}$$

$$\Delta \phi = \phi(x, y, z) - \phi(x_0, y_0, z_0)$$

For a small change in position along the direction of $\hat{u}$

$$\Delta \phi \approx \frac{d\phi}{ds} ds$$

$$\frac{d\phi}{ds} = ?$$
Potential

Let

\[ \mathbf{\hat{u}} = a\mathbf{x} + b\mathbf{y} + c\mathbf{z} \]
\[ a^2 + b^2 + c^2 = 1 \]
\[ \Delta \mathbf{r} = \mathbf{r} - \mathbf{r}_0 = s\mathbf{\hat{u}} \quad \implies \quad \mathbf{r} = \mathbf{r}_0 + s\mathbf{\hat{u}} \]
\[
\begin{cases}
    x = x_0 + as \\
    y = y_0 + bs \\
    z = z_0 + cs
\end{cases}
\]
\[ \phi(x, y, z) = \phi[x(s), y(s), z(s)] \]

\[ \frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} \frac{dx}{ds} + \frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial z} \frac{dz}{ds} \]
\[ = a \frac{\partial \phi}{\partial x} + b \frac{\partial \phi}{\partial y} + c \frac{\partial \phi}{\partial z} \]
\[ = (a\mathbf{x} + b\mathbf{y} + c\mathbf{z}) \cdot \left( \mathbf{x} \frac{\partial \phi}{\partial x} + \mathbf{y} \frac{\partial \phi}{\partial y} + \mathbf{z} \frac{\partial \phi}{\partial z} \right) \]
\[ \frac{d\phi}{ds} = \nabla \phi \cdot \mathbf{\hat{u}} \]
\[ \nabla \phi = \mathbf{\hat{x}} \frac{\partial \phi}{\partial x} + \mathbf{\hat{y}} \frac{\partial \phi}{\partial y} + \mathbf{\hat{z}} \frac{\partial \phi}{\partial z} = \text{grad} \phi \]
What is $\nabla \phi$?

$$\frac{d\phi}{ds} = \nabla \phi \cdot \hat{u} = |\nabla \phi| \cos \theta$$

The largest value of $d\phi/ds$ is found in the direction of $\nabla \phi$!
What is $\nabla \phi$?

Consider an equipotential surface:

$\phi_P = \phi_A = \phi_B = \phi_C$

$\Delta \phi = 0 \rightarrow \frac{\Delta \phi}{\Delta s} = 0$

When

$\Delta s \rightarrow 0$

$PA \rightarrow PB \rightarrow PC \rightarrow \hat{u}$

$\frac{\Delta \phi}{\Delta s} \rightarrow \frac{d\phi}{ds} = 0$ along $\hat{u}$

$\nabla \phi \cdot \hat{u} = |\nabla \phi| \cos \theta = 0 \implies \nabla \phi \perp \hat{u}$

$\nabla \phi$ is perpendicular to the surface $\phi = \text{constant}$. 

Gravitational Potential

\[ U(z) = gz \]
\[ \nabla U = g\hat{z} \]

Equipotential:
\[ U(z) = gz = \text{const.} \]
\[ z = \text{const.} \]

An equipotential surface \((U = \text{constant})\) is a plane parallel to the \(x-y\) plane.

\(dU/ds\) is the largest in the direction of \(\hat{z}\).

\[ \nabla U = g\hat{z} = -\vec{g} \]

\[ \vec{g} = -\nabla U \]
\[ \vec{F} = m\vec{g} \]
\[ \vec{n} \phi \text{ in Different Coordinate Systems} \]

\[ \nabla U \text{ in Cartesian coordinates} \]

\[ \nabla U = \hat{x} \frac{\partial U}{\partial x} + \hat{y} \frac{\partial U}{\partial y} + \hat{z} \frac{\partial U}{\partial z} \]

\[ \nabla U \text{ in polar coordinates} \]

\[ \nabla U = \hat{e}_r \frac{\partial U}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial U}{\partial \theta} \]

\[ \nabla U \text{ in cylindrical coordinates} \]

\[ \nabla U = \hat{e}_r \frac{\partial U}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial U}{\partial \theta} + \hat{e}_z \frac{\partial U}{\partial z} \]

\[ \nabla U \text{ in spherical coordinates} \]

\[ \nabla U = \hat{e}_r \frac{\partial U}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial U}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \]
Point Charge

Electrostatic potential:

\[ V(r) = \frac{kq}{r} \]

\[ \nabla V = \hat{e}_r k q \frac{\partial}{\partial r} \left( \frac{1}{r} \right) = -\frac{kq}{r^2} \hat{e}_r \]

Equipotential surface:

\[ r = \text{const.} \]

\[ \nabla V = -\vec{E} \]

\[ \vec{F} = q \vec{E} \]