PC1134 Lecture 1

Topic

• Review of differentiation

Objectives

To become familiar with

• ordinary differentiation

• basic techniques of differentiation

• derivatives of common functions

Relevance

• These concepts and techniques are important for the study of partial differentiation.
Function

what is a function?

In mathematics, especially in its applications to physical science, we are often interested in the relations and connections between different numbers or sets of numbers. A *function* is a way of expressing such a connection.

\[ y = f(x) \]

\(x\) is referred as *independent variable*, \(y\) is the *dependent variable*. 
Examples of function

- *Position* of an moving object can be a function of *time*.
  \[ x(t) = x_0 + vt + \frac{1}{2}at^2 \]

- *Potential* of a point charge is a function of *r*, distance from the point charge.
  \[ V(r) = \frac{1}{4\pi\varepsilon} \frac{Q}{r} \]

- The *temperature* of an ideal gas is a function of pressure for *constant* volume
  \[ PV = nRT \]

  where \( n \) and \( R \) are referred as constants or parameters
Representing Function

A function can be represented by

• an analytic \textbf{equation}

\[ V = \frac{1}{2} k x^2 \]

• a \textbf{table} (for discrete variable)

<table>
<thead>
<tr>
<th>Time</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8:00</td>
<td>25.0</td>
</tr>
<tr>
<td>9:00</td>
<td>26.2</td>
</tr>
<tr>
<td>10:00</td>
<td>27.5</td>
</tr>
<tr>
<td>11:00</td>
<td>28.7</td>
</tr>
<tr>
<td>12:00</td>
<td>30.0</td>
</tr>
<tr>
<td>13:00</td>
<td>31.4</td>
</tr>
<tr>
<td>14:00</td>
<td>32.6</td>
</tr>
<tr>
<td>15:00</td>
<td>32.0</td>
</tr>
<tr>
<td>16:00</td>
<td>31.3</td>
</tr>
<tr>
<td>17:00</td>
<td>30.8</td>
</tr>
</tbody>
</table>
Representing Function (cont.)

- or a graph
Derivative

Given $y = f(x)$, a change in $x$ will cause a change in $y$.

$$x \rightarrow x + \Delta x$$
$$y \rightarrow y + \Delta y$$

How is $\Delta y$ related to $\Delta x$?

$$y = f(x)$$

$$y + \Delta y = f(x + \Delta x)$$

$$\Delta y = f(x + \Delta x) - f(x)$$

"Rate" of change

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

When $\Delta x \rightarrow 0$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = y'$$
Derivative

\[ \frac{dy}{dx} \] is the rate of change of \( y \) with respect to \( x \).

Graphically, the value of \( \frac{dy}{dx} \) at any particular value of \( x \) is equal to the gradient of the tangent to the graph of \( y \) against \( x \) at that particular value of \( x \).
Higher Derivatives

In general, \( \frac{dy}{dx} \) is a function of \( x \) and can be written as \( y'(x) \), \( f'(x) \) or \( \frac{dy}{dx}(x) \).

Let

\[
z(x) = f'(x)
\]

Since \( z(x) \) is a function of \( x \), we can calculate its derivative

\[
\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}
\]

This is called the second derivative of \( y \) with respect to \( x \) and is written as

\[
\frac{d^2y}{dx^2}, \text{ or } y'' \text{ or } f''(x)
\]

Higher derivatives can be defined similarly.
Rules

- If \( y = f(x) \pm g(x) \)
  then \( y' = f' \pm g' \)

- If \( y = f(x)g(x) \)
  then \( y' = fg' + f'g \)

- If \( y = \frac{f(x)}{g(x)} \)
  then \( y' = \frac{f'g - fg'}{g^2} \)
Chain Rule

Function of a function:

\[ y = y(x) \]
\[ x = x(t) \]

\[ \frac{dy}{dt} = ? \]

\[ \frac{\Delta y}{\Delta t} = \frac{\Delta y \Delta x}{\Delta x \Delta t} \]

\[ \Rightarrow \]

\[ \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \]
Inverse Function

\[ y = f(x) \implies x = g(y) \]

\[ \frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}} \]

\[ \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \]
Implicit Differentiation

Consider the function defined by the equation

\[ x^3 - 3xy + y^3 = 2 \]

It cannot be represented explicitly in the form of

\[ y = f(x) \]

We can differentiate term by term with respect to \( x \)

\[
\frac{d}{dx} (x^3) - \frac{d}{dx} (3xy) + \frac{d}{dx} (y^3) = \frac{d}{dx} (2)
\]

\[ \Rightarrow 3x^2 - \left( 3x \frac{dy}{dx} + 3y \right) + 3y^2 \frac{dy}{dx} = 0 \]

Rearranging terms gives

\[
\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}
\]
Function Defined Parametrically

Given

\[ y = y(t) \]
\[ x = x(t) \]

What is \( \frac{dy}{dx} \)?

\[ \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \frac{\Delta t}{\Delta x} \]

\[ \frac{dy}{dx} = \frac{dy}{dt} \left/ \frac{dx}{dt} \right. \]
Derivatives of Elementary Functions

Algebraic function

\[ y = x^r \quad \frac{dy}{dx} = r x^{r-1} \]

Trigonometrical function

\[ \frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x \]

\[ \frac{d}{dx} \tan x = \frac{1}{\cos^2 x} \quad \frac{d}{dx} \cot x = -\frac{1}{\sin^2 x} \]

Inverse trigonometrical function

\[ \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \quad \frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2} \]
Derivatives of Elementary Functions

Exponential function

\[ y = e^x \quad \frac{dy}{dx} = e^x = y \]

Logarithmic function

\[ \frac{d}{dx} \ln x = \frac{1}{x} \quad \frac{d}{dx} \log_a x = \frac{1}{x \ln a} \]

Hyperbolic functions

\[ \sinh x = \frac{1}{2} (e^x - e^{-x}) \quad \frac{d}{dx} \sinh x = \cosh x \]

\[ \cosh x = \frac{1}{2} (e^x + e^{-x}) \quad \frac{d}{dx} \cosh x = \sinh x \]

\[ \tanh x = \frac{\sinh x}{\cosh x} \quad \frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} \]

\[ \coth x = \frac{\cosh x}{\sinh x} \quad \frac{d}{dx} \coth x = \frac{1}{\sinh^2 x} \]
Derivatives of Elementary Functions

Inverse hyperbolic functions

\[
\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}
\]

\[
\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}
\]

\[
\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2} = \frac{d}{dx} \coth^{-1} x
\]
Example 1

\[ y = \sin^{-1} \left( 2x \sqrt{1 - x^2} \right) \]

Let 
\[ y = \sin^{-1} z \text{ and } z = 2x \sqrt{1 - x^2} \]
then
\[ \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} \]

\[ \frac{dy}{dz} = \frac{d}{dz} \sin^{-1} z = \frac{1}{\sqrt{1 - z^2}} \]

Treating \( 2x \sqrt{1 - x^2} \) as the product of two function \( 2x \) and \( \sqrt{1 - x^2} \)

\[ \frac{dz}{dx} = \frac{d}{dx} (2x) \cdot \sqrt{1 - x^2} + 2x \frac{d}{dx} \sqrt{1 - x^2} \]
**Example 1**

The derivative of $\sqrt{1 - x^2}$ can be evaluated by apply the chain rule again. The result is

$$\frac{d}{dx} \sqrt{1 - x^2} = -\frac{x}{\sqrt{1 - x^2}}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - z^2}} 2\sqrt{1 - x^2} + 2x \frac{-x}{\sqrt{1 - x^2}}$$

$$= \frac{1}{\sqrt{1 - 4x^2(1 - x^2)}} \frac{2(1 - 2x^2)}{\sqrt{1 - x^2}}$$

$$= \pm \frac{2}{\sqrt{1 - x^2}}$$

*Where does the ± come from?*
Example 2

If \( y = x^5 + x \), find \( \frac{d^2x}{dy^2} \).

\[
\frac{dy}{dx} = 5x^4 + 1
\]

\[
\frac{dx}{dy} = \frac{1}{5x^4 + 1}
\]

\[
\frac{d^2x}{dy^2} = \frac{d}{dy} \left( \frac{1}{5x^4 + 1} \right)
\]

\[
= \frac{d}{dx} \left( \frac{1}{5x^4 + 1} \right) \frac{dx}{dy}
\]

\[
= -\frac{20x^3}{(5x^4 + 1)^2} \cdot \frac{1}{5x^4 + 1}
\]

\[
= -\frac{20x^3}{(5x^4 + 1)^3}
\]
Example 3

\[ x = a \cos^3 t, \quad y = a \sin^3 t \]

\[ \frac{d^2y}{dx^2} = ? \]

\[ \frac{dy}{dx} = \frac{dy}{dt} \left/ \frac{dx}{dt} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\tan t \right. \]

\[ \frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan t) = \frac{d}{dt}(-\tan t) \frac{dt}{dx} \]

\[ = -\frac{1}{\cos^2 t} (3a \cos^2 t \sin t)^{-1} \]

\[ = \frac{1}{3a \cos^4 t \sin t} \]