The quantum finance pricing formulas for coupon bond options and swaptions derived by Baaquie [Phys. Rev. E 75, 016703 (2006)] are reviewed. We empirically study the swaption market and propose an efficient computational procedure for analyzing the data. Empirical results of the swaption price, volatility, and swaption correlation are compared with the predictions of quantum finance. The quantum finance model generates the market swaption price to over 90% accuracy.

**I. INTRODUCTION**

A swaption is an interest rate option that has a deep and liquid market, and arguably is even the most liquid option on interest rates that is available today. Participants in the swaption market tend to focus their attention on the highly liquid European swaptions, that is, swaptions that can be exercised only on a single date. The theoretical price of these instruments depends on the forward interest rates model being used.

In quantum finance the forward interest rates are modeled as a two-dimensional quantum field theory [2] and has shown great advantage in pricing and hedging interest rate options [3,4]. This encourages us to investigate the swaption market in some detail within the quantum finance framework. Swaptions can be expressed as a special case of coupon bond options. Swaption pricing is a nonlinear problem which usually needs to be studied numerically. However, as shown in Baaquie [1], henceforth called paper I, Feynman perturbation expansion gives a closed form approximation for the price and correlation of swaptions. All the pricing formulas and empirical results for swaptions obtained in this paper are also valid for the coupon bond option.

In Sec. II the relevant formulas for swaption price, volatility, and correlation are reviewed and the computational framework is defined. In Sec. III the data of swaptions and zero coupon yield curve are discussed in some detail. Data processing and computational procedures are discussed in Sec. IV with some empirical results being given in Sec. V.

**II. PRICE AND CORRELATION OF SWAPTIONS**

We review the results, given in paper I, on swaption price, volatility, and correlation that provide the analytical basis for our empirical study.

Swaps and swaptions are financial instruments that are widely used in the debt market. Swaps are interest rate derivatives in which one party pays floating interest rate, determined by the prevailing Libor at the time of the payment, with the other party paying at a prefixed interest rate $R_S$. The swaptions that we are studying have floating interest rate payments that are paid at $\ell \equiv 3$ month intervals and fixed rate payments that are paid at intervals of $2\ell \equiv 6$ months. The three monthly floating rate payments are paid at times $T_0+n\ell$, with $n=1,2,\ldots,N$; there are $N$ payments. For six monthly fixed rate payments there are only $N/2$ payments of amount $2R_S$, made at times $T_0+2n\ell$, $n=1,2,\ldots,N/2$.

The payoff function for the interest rate swaption, in which the holder of the option receives at the fixed rate and pays at the floating rate, is given in paper I by [5]

$$C(T_0;R_S) = V \left( B(T_0,T_0+N\ell) + 2 \ell R_S \sum_{n=1}^{N/2} B(T_0,T_0+2n\ell) - 1 \right),$$

where $B(t,T)$ is the price of a zero coupon Treasury Bond at time $t$ that matures at time $T>t$. The coefficients and strike price for a swaption are hence given by

$$c_n = 2\ell R_S, \quad n=1,2,\ldots,(N-1)/2,$$

$$payment \ at \ T_0+2n\ell,$$

$$c_{N/2} = 1+2\ell R_S, \quad payment \ at \ T_0+N\ell,$$

$$K=1.$$

The fixed interest rate par value $R_p$, at time $t_0$, is such that the value of the interest rate swap has zero value. Hence

$$\sum_{n=1}^{N/2} c_n B(t_0,T_0+2n\ell) - 1 = 0 \Rightarrow 2\ell R_p(t_0) = \frac{B(t_0,T_0+N\ell) - B(t_0,T_0+2N\ell)}{\sum_{n=0}^{N/2} B(t_0,T_0+2n\ell)}.$$

The price of a coupon bond option $C(t_0,t,R_S)$ at time $t_0 < t$, using the money market measure and discounting the value of the payoff function using the spot interest rate $r(t) = f(t,t)$, is given from Eq. (1) by

\[ 1 \text{Suppose the swaption has a duration such that } N \text{ is even. Note } N=4 \text{ for a year long swaption.} \]
where $V$ is the notional deposit on which the interest is calculated; we set $V=1$.

The option price has been derived in paper I and yields the results given below:

$$
C(t_0, t_*, R_S) = B(t_0, t_*) \left[ e^{-i\omega(\text{str})} \sum_{n=1}^{N/2} c_n B(t_*, t_* + 2n \ell) - 1 \right] e^{(1/2)X^2},
$$
\label{eq:option_price}

(4)

the other correlators that are required for the computation of swaption volatility.

The input data that we need for computing the swaption price can be derived from the underlying forward interest rates’ data and yield the coupon bond price, the forward bond price, and the fixed rate par value $R_P$.

A. Swaption at the money

Recall the par value of the fixed interest rate $R_p(t_0)$ is the value for the fixed interest payments for which the swap at time $t_0$ is zero. From Eqs. (8) and (3) fixed interest rate equal to par value, namely $R_S=R_p$ implies the following:

$$
F = F(t_0) = \sum_{i=1}^{N/2} c_i F(t_0, T_i, T_0) + 2i \ell,
$$
\label{eq:swaption_at_money}

(5)

$$
F(t_0, T_0, T_0 + 2i \ell) = B(t_0, T_0) - B(t_0, T_0 + N \ell) + \sum_{n=0}^{N/2} F(t_0, T_0 + 2n \ell) + F(t_0, T_0 + N \ell) = 1.
$$
\label{eq:swaption_at_money_2}

(10)

In the coupon bond option pricing formula $X = (F-K)/\sqrt{A}$ and for swaptions $K=1$. Hence when the fixed interest rate $R_S$ for the swaption is at the money $F=1$ and this leads to $X = (F-K)/\sqrt{A}=0$. As discussed in paper I, the asymptotic behavior of the error function yields the following:

$$
I(X) = 1 - \sqrt{\frac{\pi}{2}}X + O(X^2), \quad X \approx 0 \quad \text{and hence the swaption close to the money, to leading order, has the form}
$$

$$
C(t_0, t_*, R_p) = B(t_0, t_*) \sqrt{\frac{A}{2\pi}} - \frac{1}{2} B(t_0, t_*) (K - F) + O(X^2).
$$
\label{eq:swaption_at_money_3}

(12)

B. Volatility and correlation of swaptions

The volatility and correlation of swaption prices are important quantities since they are indicators of the market’s direction and also give us insights on portfolio study.

Consider the volatility and correlation of the change of swaption price for infinitesimal time steps. Let $C_1 = C(t_0, t_1, R_1)$ and $C_2 = C(t_0, t_2, R_2)$ denote two swaptions. Denote time derivative by an upper dot; for infinitesimal time step $\epsilon$ we have

$$
\langle \dot{C}_1 \dot{C}_2 \rangle = \frac{1}{\epsilon} \langle \{ C_1(t_0 + \epsilon) - C_1(t_0) \} \{ C_2(t_0 + \epsilon) - C_2(t_0) \} \rangle = \frac{1}{\epsilon} \langle [\delta C_1(t_0) \delta C_2(t_0)] \rangle,
$$
\label{eq:volatility}

(13)

where the connected correlator is defined by $\langle AB \rangle_c = \langle AB \rangle - \langle A \rangle \langle B \rangle$. 

\[ \text{016704-2} \]
Note the swaption prices $C_1, C_2$ depend on the forward bond prices $F$, which take random values every day. The random changes in the price of the forward bond prices lead to changes in the price of the swaption. The correlation function $\langle \partial C_1(t_0) \partial C_2(t_0) \rangle_c$ can be evaluated by a historical average over the daily swaption prices, considered as the random outcomes of the swaption price due to the random changes in the forward bond price. Hence a historical average of the correlator of changes in the swaption price can be equated to the ensemble average of the correlator taken over the random fluctuations of the forward bond prices.

The field theory of forward interest rates yields, from paper I, that

$$\langle \hat{f}(t,x) \hat{f}(t,x') \rangle_c = \frac{1}{\epsilon} M(t,x,x').$$

(14)

From the pricing formula given in Eq. (12), the swaption’s rate of change at the money, namely $X=0$, is given by

$$\sqrt{2\pi} d\mathbb{C}(t_0, t_1, R_p) = \frac{dB(t_0, t_1)}{dt_0} - \frac{1}{2} \frac{dA_1}{dt_0} + \sqrt{\frac{\pi}{2}} \frac{dB(t_0, t_1)}{dt_0}$$

$$= D_1 - C(t_0, t_1, R_p) \int_{t_0}^{T_1} df(t_0, x) - \frac{B(t_0, t_1)}{A_1} \sum_{i,j=1}^{N} \int_{t_0}^{T_1} df(t_0, x) - \frac{\sqrt{\pi}}{2} B(t_0, t_1) \sum_{i=1}^{N} \int_{t_0}^{T_1} df(t_0, x),$$

(15)

where $t_1$ denotes $t_1$ or $t_2$ and $D_1$ contains all the deterministic (nonstochastic) factors that are subtracted out in forming the connected correlation functions.

To determine $\hat{C}$, as seen from the equation above, one needs $\hat{f}(t_0, x)$, namely the evolution equation of the quantum field $f(t,x)$, and is discussed in paper I. The evolution equation of $f(t,x)$, together with Eqs. (14) and (15), yields the following:

$$2\pi \epsilon (\partial C_1(t_0) \partial C_2(t_0))_c = C_1 C_2 \int_{t_0}^{T_1} dx \int_{t_0}^{T_1} dx' M(t_0, x, x')$$

$$+ \frac{B(t_0, t_1)}{\sqrt{A_1}} C_2 \sum_{i,j=1}^{N} \int_{t_0}^{T_1} dx' \int_{t_0}^{T_1} dx' M(t_0, x, x')$$

$$+ \frac{B(t_0, t_1)}{\sqrt{A_2}} C_1 \sum_{j=1}^{N} \int_{t_0}^{T_1} dx' \int_{t_0}^{T_1} dx' M(t_0, x, x')$$

$$+ \sqrt{\frac{\pi}{2}} B(t_0, t_1) C_1 \sum_{i,j=1}^{N} \int_{t_0}^{T_1} dx' \int_{t_0}^{T_1} dx' M(t_0, x, x')$$

$$+ \sqrt{\frac{\pi}{2}} B(t_0, t_1) C_2 \sum_{i,j=1}^{N} \int_{t_0}^{T_1} dx' \int_{t_0}^{T_1} dx' M(t_0, x, x')$$

$$+ \sqrt{\frac{\pi}{2}} B(t_0, t_1) B(t_0, t_2) \sum_{i,j=1}^{N} \int_{t_0}^{T_1} dx' \int_{t_0}^{T_1} dx' M(t_0, x, x')$$

(16)

where $A_1$ and $A_2$ denote $A$ for the two swaptions, respectively.

**C. Market correlator**

The forward bond price correlator $G_{ij}$, the swaption correlator, and volatility are all computed from a set of three-dimensional integrations on $M(t,x,x')$ with various integration limits. A general form of all the integration is given as follows:

$$I = \int_{t_0}^{m_1} dt \int_{m_2}^{x_1} dx \int_{m_3}^{x_2} dx' M(t,x,x').$$

(17)

and the limits of integrations are listed in Table I.
III. DATA FROM SWAPTION MARKET

The swaption market provides daily data for \( X \) by \( Y \) swaptions. These swaptions mature \( X \) years from today, with the underlying swap starting at time \( X \) and the last payment being paid \( X+Y \) years in the future. The domain for the swaption instrument is given in the time and future time plane in Fig. 1.

All the prices are presented with interest rates in basis points (100 basis points = 1% annual interest rate) and must be multiplied by the notional value of one million dollars. Daily swaption prices at the money are quoted from January 29, 2003 to January 28, 2005, a total of 523 daily data. In order to get accurate results, actual days in the real 6 months are divided by 360, since the convention for total number of days in a year is 360.

**ZCYC data.** In order to generate swaption prices and swaption correlation from the model, both the historical and current underlying forward interest rates are required. The value of the coupon bond and forward bond price and the par fixed rate \( R_p \) are computed from the current forward interest rates. The integrand of the forward bond correlator \( G_{ij} \), namely \( M(t,x,x') \), is derived from historical forward interest rates’ data.

Our analysis uses Bloomberg data for the zero coupon yield curve (ZCYC), denoted by \( Z(t_0,T) \), from January 29, 2003 to January 28, 2005, and which yields, in total, 523 daily ZCYC data. The ZCYC is necessary for evaluating long duration swaptions since Libor data exist for maturity of only up to a maximum of 7 years in the future, whereas ZCYC has data with maturity of up to 30 years.

The ZCYC is given in the \( \theta = x-t = \text{constant} \) direction as shown in Fig. 2, with the interval of \( \theta \) between two data points not being a constant. Cubic spline is used for interpolating the data to a 3 month interval.

From Ref. [5] we have that the zero coupon bond is given by

\[
B(t_0,T) = \frac{1}{[1 + Z(t_0,T)/c]^{(T-t_0)c}},
\]

where \( c \) represents how many times the bond is compounded per year. For ZCYC \( c \) is given as half yearly, and hence we have \( c = 2/\text{year} \). As expected the forward bond price is given by

\[
F(t_0,t_*,T) = \frac{B(t_0,T)}{B(t_0,t_*)}.
\]

From the definition of the zero coupon bond,

\[
B(t_0,T) = \exp \left( -\int_{t_0}^{T} df(t_0,x) \right),
\]

we obtain, from Eq. (18), the following:

\[
\int_{t_0}^{T} f(t_0,x) dx = \ln[[1 + Z(t_0,T)/c]^{(T-t_0)c}].
\]
IV. NUMERICAL ALGORITHM FOR THE FORWARD BOND CORRELATOR

The market value of the forward bond price correlator \( I \) given in Eq. (14) can be derived from ZCYC data. From Eq. (14) and for discrete time \( \delta t = \delta f / \epsilon \) the correlation for changes in the forward interest rates is given by [2]

\[
M(t, x, x') = \frac{1}{\epsilon} \langle \delta f(t, x) \delta f(t, x') \rangle_c,
\]

\[
\delta f(t, x) = f(t + \epsilon, x) - f(t, x).
\]  

(21)

Thus, we have for the forward bond correlators the following:

\[
I = \epsilon \int_{t_0}^{t_1} dt \int_{m_2}^{d_1} dx \int_{m_3}^{d_2} dx' \langle \delta f(t, x) \delta f(t, x') \rangle_c.
\]  

(22)

From Table I we see that none of the limits on the integrations over \( x, x' \) depend on the time variable \( t \); hence the finite time difference operator \( \delta \) can be moved out of the \( x, x' \) integrations and yields

\[
I = \epsilon \int_{t_0}^{t_1} dt \left( \frac{\int_{m_2}^{d_1} dx f(t, x)}{\int_{m_3}^{d_2} dx' f(t, x')} \right) \langle \delta f(t, x) \delta f(t, x') \rangle_c.
\]  

(23)

We keep to the \( x \) and \( x' \) integration variables instead of changing them to \( \theta \) and \( \theta' \) since, as discussed earlier, ZCYC data directly yield the integrals of forward interest rates on future time \( x \). The numerical values of \( \int_{m_2}^{d_1} dx f(t, x) \) and \( \int_{m_3}^{d_2} dx' f(t, x') \) are obtained from the market values of the ZCYC.

To evaluate the market correlator \( I \) one needs to know the value of the correlator \( M(t, x, x') \) in the future; the reason being that the time integration \( t \) in \( I \) runs from present time \( t_0 \) to time \( m_1 > t_0 \) in the future. The problem of obtaining the future values of \( M(t, x, x') \) can be solved by assuming that the correlation function for changes in the forward interest rates is invariant under time translations; that is

\[
M(t, x, x') = M(t - a, x - a, x' - a).
\]  

(24)

The assumption of time translation invariance of the forward rates correlation function has been empirically tested in Ref. [6]; although this assumption cannot be indefinitely extended, a 2 year shift is considered to be reasonable [6].

The integration on the \( t \) axis can be converted to a summation by discretizing time into a lattice with spacing \( \epsilon' \); one then obtains

\[
I = \epsilon' \sum_{t_k}^{m_1-t_0} \int_{m_2}^{d_1} dx \int_{m_3}^{d_2} dx' M(t_0 + t_k, x, x').
\]  

(25)

From Eq. (25) and a change of variables yields

\[
x = y + t_k, \quad x' = y' + t_k.
\]

We hence have from Eq. (25),

\[
I = \epsilon' \sum_{t_k}^{m_1-t_0} \int_{m_2}^{d_1} dy \int_{m_3}^{d_2} dy' M(t_0 + t_k, y + t_k, y' + t_k)
\]

\[
= \epsilon' \sum_{t_k}^{m_1-t_0} \int_{m_2}^{d_1} dy \int_{m_3}^{d_2} dy' M(t_0, y, y'),
\]  

(26)

where the condition given in Eq. (24) has been used to obtain Eq. (26). The integration on future data has been replaced by a summation on the current value of \( M(t_0, x, x') \), with \( x, x' \) taking values on various intervals. The current value of \( M(t_0, x, x') \) in turn is evaluated by taking averages of the correlator over its past values.

From above and Eqs. (21), (23), and (26) we have

\[
I = \epsilon' \sum_{t_k}^{m_1-t_0} \left( \int_{m_2}^{d_1} dy \int_{m_3}^{d_2} dy' \langle \delta f(t_0, y) \delta f(t_0, y') \rangle_c \right)
\]

\[
= \epsilon' \sum_{t_k}^{m_1-t_0} \left( \int_{m_2}^{d_1} dy \int_{m_3}^{d_2} dy' \langle \delta f(t_0, y) \delta f(t_0, y') \rangle_c \right).
\]

(27)

As discussed earlier, in order to directly use the ZCYC data the finite time difference operator \( \delta \) is taken outside the future time integrations. Note \( \epsilon' \) is the time integration interval and is equal to \( \epsilon \) for the time summation with daily intervals \( \epsilon = \epsilon' = 260 \) (260 is the actual number of trading days in one year).

Reexpressing \( I \) in terms of the ZCYC data we obtain

\[
I = \epsilon' \sum_{t_k}^{m_1-t_0} \langle \delta Y(t_0, m_2 - t_k, d_1 - t_k) \delta Y(t_0, m_3 - t_k, d_2 - t_k) \rangle_c.
\]

(28)

where, from Eq. (20), we have

\[
Y(t_0, t, T) = \int_{t}^{T} f(t_0, x) dx = \int_{t_0}^{T} f(t_0, x) dx - \int_{t_0}^{t} f(t_0, x) dx
\]

\[
= \ln[(1 + Z(t_0, T)/c)^{(T-t_0)/c}] - \ln[(1 + Z(t_0, t)/c)^{(T-t_0)/c}].
\]  

(29)

The forward bond price correlator’s present value (at time \( t_0 \)) is obtained by averaging the value of the correlator \( \langle \delta Y(t_0, m_2 - t_k, d_1 - t_k) \delta Y(t_0, m_3 - t_k, d_2 - t_k) \rangle \) over the last \( t_0 - t_A \) days with \( t_A = 180 \) days.\(^3\) Since the computation requires the value of \( \delta Y \) for different future time intervals \( x, x' \) we must again use cubic splines to interpolate ZCYC for obtaining daily values of the ZCYC. The shift of the future time integration to the present and the domain used for doing the averages for the correlator is illustrated in Fig. 3.

V. EMPIRICAL RESULTS

The \( 2 \times 10 \) and \( 5 \times 10 \) swaptions are priced for time series April 6, 2004–January 28, 2005 using the pricing formula from Sec. II. When computing the forward interest rates’

\(^3\)We ran the program by adding 30 days to the time averaging for evaluating the expectation values of the correlators; the best fit is given when the averaging is done over the past 180 days; see Sec. V.
correlator $M(t;x,x')$ we found that daily swaption prices are stable when more than 270 days of historical data for ZCYC are used; but a 270-day average does not give the best fit of the predictions of model swaption price with the swaption’s market value. This may be due to too much old information creating large errors in the predictions for the present day swaption prices. However, averaging on less historical data causes the swaption price curve to fluctuate strongly since it is likely that new information dominates swaption pricing and makes the price too sensitive to small changes.

Our empirical studies and results show that a moving averaging of 180 days of historical data gives the best result for this period. One can most likely improve the accuracy by higher frequency sampling of 180 days of historical data.

The results obtained from the field theory model are compared with daily market data and are shown in Figs. 4 and 5, with normalized root mean square of error being 3.31% and 6.31%, respectively.

The results for the swaption volatility and correlation discussed in Sec. II B are derived for the change on the same instruments; from Eq. (22)

<table>
<thead>
<tr>
<th>$G_{ij}$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$d_1$</th>
<th>$d_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{ij}$</td>
<td>$t_*$</td>
<td>$t_*$</td>
<td>$t_*$</td>
<td>$T_1$</td>
<td>$T_2$</td>
</tr>
<tr>
<td>$G_{ij}$</td>
<td>$t_1$</td>
<td>$t_1$</td>
<td>$t_1$</td>
<td>$T_1$</td>
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<td>$G_{ij}$</td>
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<td>$t_2$</td>
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</tr>
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</table>

FIG. 3. Shaded area $A$ is the integration domain of $I$. For the case when $t=t_0+t_k$, the integration of $x$ and $x'$ for evaluating the expression for $I(t_0+t_k,t,T)$ inside $\int dY(t_0+t_k,t,T)\int dY(t_0$ $+t_k,t,T)$ is shifted back to $t_0$. Invariance in time yields this to be equal to $\langle \delta Y(t_0,t_0,t_0,T)\rangle$. A historical average is done over the shaded area $B$, which is in the past of $t_0$, $t_k$ =180 days is the optimum number of past data for evaluating the historical averages.

[The rest of the text continues with further details and figures related to the analysis and results.]
study swaption pricing in the HJM model. By considering the volatility function to have the special form of $\sigma(t,x) = \sigma_0 e^{-\lambda(t-x)}$ Jarrow and Turnbull [5] obtained, for the one-factor HJM model, the following explicit expression for the coupon bond option:

$$C_{\text{HJM}}(t_0, t_s, K) = \sum_{i=1}^{N} c_i B(t_0, T_i) N(d_i) - KB(t_0, t_s) N(d),$$

$$d_i = \frac{r'}{\sigma_R} + W(t_s, T_i) \sigma_R,$$

$$W(t_s, T_i) = \frac{1}{\lambda} (1 - e^{-\lambda(T_i-t_s)}), \quad \sigma^2 = \frac{\sigma_0^2}{2\lambda} (1 - e^{-2\lambda(t-t_0)}).$$

The quantity $r'$ is related to the strike price $K$ by a nonlinear transformation that depends on the initial coupon bond price $\sigma_0$. As shown in paper I, to leading order in $\sigma_0$ the HJM limit of the field theory pricing formula with exponential volatility given by $\sigma(t,x) = \sigma_0 \exp[-\lambda(x-t)]$ yields the HJM pricing formula.

We estimate $\sigma_0$ for the exponential volatility function in the HJM model from historical ZCYC data. By using exponential volatility and daily forward bond prices obtained from ZCYC, we price the swaption with the HJM pricing formula and in Fig. 7 compare it with the market price and the field theory price.

The results show that the HJM model is inadequate for pricing swaptions, both because it systematically overprices the swaption by a large amount, and also because the instability of the price itself would give incorrect results if one tries to hedge the swaption using the one-factor HJM pricing formula.

Instead of using the HJM formula for pricing the coupon bond options practitioners may consider representing the price of the swaption by an implied volatility using the HJM pricing formula. However, unlike the case for the price of caps where this procedure is possible [3], the entire swaption curve cannot be fitted by adjusting only one quantity $\sigma_0$. Furthermore, the implied volatility $\sigma(t,x)$ in the first place may not be able to fit the price of all swaptions, and second, it will depend on time; it is quite impractical to numerically evaluate daily implied volatility from daily swaption prices.

VI. CONCLUSION

The quantum finance swaption pricing formula was empirically tested, for various durations, by comparing its predictions with the market values. There is over 90% agreement of the theoretical predictions for the swaption’s price with its market value, with errors around 6% for most swaptions and with an accuracy of about 3% for the shorter maturity swaptions.

A comparison of the field theory model and the industry standard HJM model shows that the field theory model gives a more accurate and stable result than the HJM model.

The HJM model is not suited for pricing swaptions because the volatility parameter that goes into the pricing formula cannot be extracted from the swaption data. In contrast since the field theory model directly uses the market correlator $M(t, x, x')$ all the market information is fully accounted for in the swaption price.

The correlation of different swaptions and their volatilities are central ingredients in forming swaption portfolios and hedging these portfolios. The quantum finance swaption pricing formula provides an approximate analytic result that can in turn be used to compute the correlation and volatility of swaptions; based on these analytic results one can form and hedge interest rate portfolios.
ACKNOWLEDGMENTS

The authors thank Jiten Bhanap for many useful discussions, and for explaining to the authors the intricacies of swaption data. The data for our empirical studies were generously provided by Bloomberg, Singapore.

APPENDIX: TEST OF ALGORITHM FOR COMPUTING $\mathcal{I}$

The computation of $\mathcal{I}$ is the key step in calculating swaption prices. We test the program used for numerically computing $\mathcal{I}$ by an analytically solvable formula of the forward interest rates.

Consider an analytical formula for forward interest rates as given below

$$f(t,x) = 1 - e^{-\lambda(x-t)}.$$ (A1)

The forward interest rates have an exponential form and increase from $f(t,0) = 0$ to the maximum value 1. Furthermore, $f(t,x)$ depends only on $x-t$, which is what we need for carrying out the shift of time as explained in Eq. (24).

Since we can analytically perform the integration of forward interest rates, one can directly determine $Y$. The analytic expression for the $Y$ is given by

$$Y = Y(t_s, T_s)$$
$$= \int_{t_s}^{T_s} dx f(t,x)$$
$$= T_s - t_s + \frac{1}{\lambda} e^{\lambda(T_s - t_s)}.$$ (A2)

The input data are generated from our test forward interest rates and processed using the same algorithm as employed in Sec. IV. Using the forward interest rates itself as input data will cause new errors since it does not directly appear in the program being checked. Note that $Y$ depends on three variables and is not suitable as input data. In analogy with Eq. (27) we form a new variable $z(t,x)$, similar to ZCYC, such that

$$Y(t_s, T_s, T_j) = z(t, T_j) - z(t, t_s),$$

$$z(t,x) = x - t + \frac{1}{\lambda} (e^{\lambda(x-t)} - 1).$$ (A3)

The function $z(t,x)$ from the above formula is used as input data since this is a starting point for the analysis of market data.

Since we are checking $\mathcal{I}$, more concretely $G_{ij}$, we have the exact analytical result

$$G_{ij} = \int_{t_0}^{t_f} dt \langle \hat{Y}(t_s, T_j) \hat{Y}(t_s, T_j) \rangle_c$$ (A4)

and from Eq. (A2) we have

$$\hat{Y}(t_s, T_j) = e^{\lambda(T_j - t_s)} - e^{-\lambda T_j} \approx b e^{\lambda T_j}. \quad (A5)$$

Changing the integration on $t$ to a summation, we have

$$G_{ij} = e^{\epsilon} \sum_{k=0}^{N} \int_{t_s}^{T_j} dx \int_{t_s}^{T_j} dx' M(x - (t_0 + k \epsilon'), x' - (t_0 + k \epsilon')),$$ (A6)

where $N=(t_s-t_0)/\epsilon'$. Note

$$e^{\epsilon} \sum_{k=0}^{N} (\hat{Y}(t_0, t_s - k \epsilon'), T_j, - k \epsilon')$$

$$\times \hat{Y}(t_0, t_s - k \epsilon', T_j, - k \epsilon'))^c,$$

and

$$\langle \hat{Y}(t_0, t_s - k \epsilon', T_j, - k \epsilon') \hat{Y}(t_0, t_s - k \epsilon', T_j, - k \epsilon') \rangle_c$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \hat{Y}(t_0 - n \epsilon', t_s - k \epsilon', T_j, - k \epsilon') \hat{Y}(t_0 - n \epsilon', t_s - k \epsilon', T_j, - k \epsilon')$$

$$- \hat{Y}(t_0 - n \epsilon, t_s - k \epsilon', T_j, - k \epsilon') \hat{Y}(t_0 - n \epsilon, t_s - k \epsilon', T_j, - k \epsilon')$$

$$\times \hat{Y}(t_0 - n \epsilon, t_s - k \epsilon', T_j, - k \epsilon')$$

$$= \frac{1}{N} b_i b_j \left( \frac{(1 - e^{-2k \epsilon \epsilon'})}{1 - e^{-2k \epsilon \epsilon}} - \frac{1}{N^2} b_i b_j \left( \frac{(1 - e^{-2k \epsilon \epsilon'})^2}{1 - e^{-2k \epsilon \epsilon^2}} \right) \right)$$ (A7)

with

$$b_i = e^{\lambda(T_i - t_0)} - e^{\lambda(T_i - k \epsilon')}.$$

We ran the program for $\mathcal{I}$ with our artificial test input and compared the result from the program with the known analytical results. The numerical test for the algorithm shows that it exactly reproduces the analytical results, verifying the correctness of the algorithm used for our empirical analysis of swaptions.