

Quantum Mechanics and Option Pricing

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Abstract

Options are financial derivatives on an underlying security. Options are important for hedging as well as speculating on future market trends. It is shown how quantum mechanics provides a natural framework for understanding options as well as for pricing these.

1 Introduction

One of the four famous papers that Einstein wrote in 1905 was on the subject of the motion of small particles suspended in a solute. This phenomenon goes under the name of Brownian motion, and points to the random movement of small particles, and is explained using the theory of random walk. In physics, the concept of white noise is introduced that generates the random paths, and is governed by the Langevin equation.

Interestingly enough, the first formalization of random walk was not in Einstein's

paper, in the study of finance. but rather in a thesis of a French graduate student, who was assigned the problem of understanding the evolution of a security, such as a stock of a organization listed with the stock market, or a bond issued by a government or a private company. To price any financial instrument one needs to assume how a stock evolves. The famous mathematician Henri Poincare assigned one of his students to study this problem.

There is a remarkable connection between the theory of random paths and the mathematical modeling in finance. The usual practise in mathematical finance is to model the random paths

2 Options on a Security

Financial derivative, or derivatives in short, are important forms of financial instruments that are traded in the financial markets. As its name implies, derivatives are **derived** from other underlying financial instruments: the cash flows of a derivative depend on the prices of the underlying instruments [1].

Derivatives have many uses from hedging a portfolio in order to reduce risk to the use of derivatives as a tool for speculation.

Given the uncertainties of the financial markets, there is a strong demand from the market for predicting the future behaviour of securities. Derivative instruments are a response for this need of the market and contain information for estimating the behaviour of a security in the future. There are three general categories of derivatives, namely forwards, futures and options.

Options are derivatives that can be written on any security, including other derivative instruments. Options are traded in the financial markets as an investor may be more interested in the profit that can be made by entering into a contract, rather than actually possessing the asset as is the case for a futures or forward contract.

An option C is a contract to buy or sell (called a put or a call) that is entered into by a buyer and seller. For example, for a **call option** the seller of the option is obliged to provide the stock of a company S at some pre-determined price K and at some fixed time in the future – if the buyer exercises the option; the buyer of the option, on the other hand has the right to either **exercise** or **not exercise** the option. If the price of the stock on maturity is less than K , then clearly the buyer of a call option should not exercise the option since he can buy the security at a lower price from the market. If, however, the price of the stock is greater than K , then the buyer makes a profit by exercising the option if it is a call. Conversely, the holder of a **put option** has the right to sell or not sell the security at a predetermined price to the seller of the put option.

An option in general is a contract with a fixed maturity, and in which the buyer has the right to – but is not obliged to – either buy or sell a security to the seller of the option at some pre-determined (but not necessarily fixed) strike price. The precise form of the strike price is called the **payoff function** of the option. There is a great variety of options, and these can be broadly classed into **path independent** and **path dependent** options.

Most of the options are traded in the derivatives market, and is a growing and highly diversified market.

Path independent options are defined by a payoff function that only depends of the

value of the underlying security at the time of maturity – the payoff function is **independent** of how the security arrived at its final price. The most widely used path independent options are the European options, and these come in two varieties, the call and the put options.

Consider an underlying security S . A European **call option** on $S(t)$, denoted by $C(t) = C(t, S(t))$, is a financial instrument that gives the owner of the option the right – but not an obligation - to buy the security at some future time $T > t_0$ for the strike price of K .

At time $t = T$, when the option matures the value of the call option $C(t, S(t))$ is clearly given by

$$C(T, S(T)) = \begin{cases} S(T) - K, & S(T) > K \\ 0, & S(T) < K \end{cases} \quad (1)$$

$$= g(S) \quad (2)$$

where $g(S)$ is the payoff function.

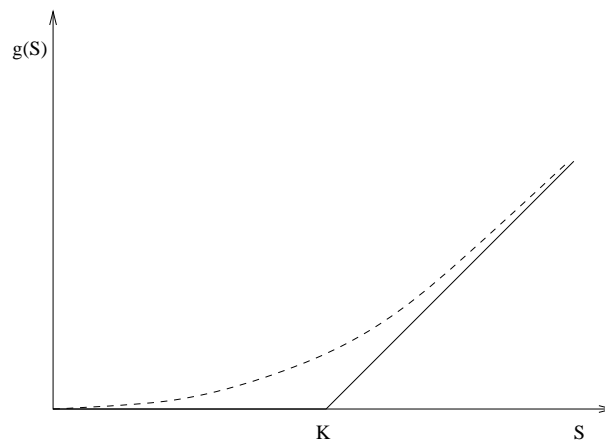


Figure 1: Payoff for Call Option; Dashed line Value of Option Before Maturity

A European put option, denoted by $P = P(t, S(t))$, gives the holder of the option the

right, but not the obligation, to sell a security S at a price of K . Clearly if the price of the security at time T is larger than the strike price K , the holder of the option will not sell the security, since he can get a better price by selling it in the market. Hence

$$P(T, S(T)) = \begin{cases} K - S(T), & K > S(T) \\ 0, & K < S(T) \end{cases} \quad (3)$$

$$= h(S) \quad (4)$$

where $h(S)$ is the payoff function of the put option.

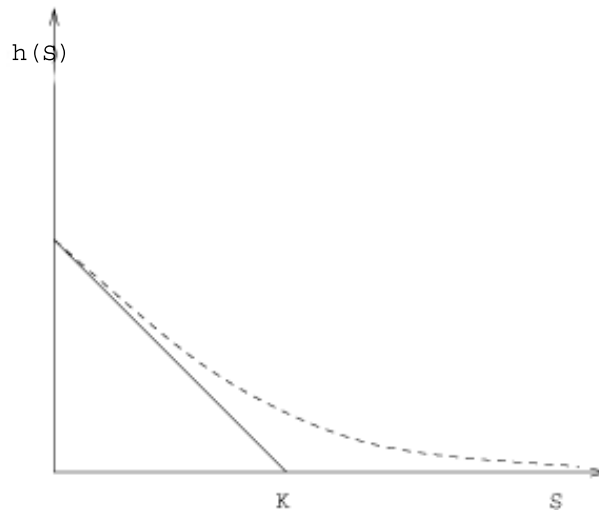


Figure 2: The payoff function $h(S)$ for a put option. The dashed line gives the value of the put option before maturity.

The concept of arbitrage is fundamental in finance, and entails that the following: there can be no risk-free return on any asset higher than the what the market has to offer. Usually the risk-free return that the market offers is the return on a fixed deposit in a bank, given by the spot interest rate $r(t)$.

For spot interest rate given by r , taken to be a constant, an argument based on the absence of arbitrage opportunities shows that

$$C(t) + Ke^{-r(T-t)} = P(t) + S(t); \quad t \leq T \quad (5)$$

and is called the put-call parity.

3 Quantum Mechanical Pricing of Options

The central problem in option pricing is the following: given the payoff function at some future time T , what is the price of the option at an earlier time $t < T$, namely $C(t, S(t))$?

The standard approach for addressing option pricing in mathematical finance is based on stochastic calculus. An independent derivation for the price of the option is given based on the formalism of quantum mechanics [2].

A stock of a company is never negative since the owner of a stock has none of the company's liabilities, and a right to dividends and pro rata ownership of a company's assets. Hence $S = e^x \geq 0$; $-\infty \leq x \leq +\infty$. The real variable x is the degree of freedom for the stock price, considered as a quantum system. The completeness equation for the degree of freedom is given by

$$\int_{-\infty}^{\infty} dx |x\rangle\langle x| = \mathcal{I} \quad : \quad \text{Completeness Equation} \quad (6)$$

where \mathcal{I} is the identity operator on (function) state space.

The option pricing is based on the following assumptions.

- All financial instruments, including the price of the option, are elements of a state space. The stock price is given by

$$S(x) = \langle x|S \rangle = e^x \quad (7)$$

The option price is given by a state vector, which for the call option price and the

payoff function is given by

$$C(t, x) = \langle x|C, t \rangle \quad (8)$$

$$g(x) = \langle x|g \rangle \quad (9)$$

and similarly for the put option.

The state space is not a normalizable Hilbert space since fundamental financial instruments such as the stock price $S(x)$ are not normalizable.

- The option price is evolved by a Hamiltonian operator H , that, due to put-call parity, evolves both the call and put options.
- The price of the option satisfies the Schrodinger equation

$$H|C, t \rangle = \frac{\partial}{\partial t}|C, t \rangle \quad (10)$$

Eq. 10 yields the following

$$|C, t \rangle = e^{tH}|C, 0 \rangle \quad (11)$$

and final value condition given in eq.(1) yields

$$|C, T \rangle = e^{TH}|C, 0 \rangle = |g \rangle \quad (12)$$

$$\Rightarrow |C, 0 \rangle = e^{-TH}|g \rangle \quad (13)$$

Hence

$$|C, t \rangle = e^{-(T-t)H}|g \rangle \quad (14)$$

or, more explicitly

$$C(t, x) = \langle x|C, t \rangle \quad (15)$$

$$= \langle x|e^{-(T-t)H}|g \rangle \quad (16)$$

Using completeness equation (6) yields

$$C(t, x) = \int_{-\infty}^{\infty} dx' \langle x | e^{-(T-t)H} | x' \rangle g(x') \quad (17)$$

$$= \int_{-\infty}^{\infty} dx' p(x, x'; \tau) g(x') \quad (18)$$

and similarly for the put option

$$P(t, x) = \int_{-\infty}^{\infty} dx' \langle x | e^{-(T-t)H} | x' \rangle h(x') \quad (19)$$

where, for $\tau = T - t$, the pricing kernel is given by

$$p(x, x'; \tau) = \langle x | e^{-\tau H} | x' \rangle \quad (20)$$

In terms of remaining time τ the pricing kernel is given by a decaying exponential as given in eq. (20).

The pricing kernel $p(x, x'; \tau)$ is the conditional probability, that given the value of the stock is e^x at time t , it will have the value of $e^{x'}$ at future time $T = t + \tau$. We see from eq. (20) that the pricing kernel is the matrix element of the differential operator $e^{-\tau H}$.

4 Hamiltonian for Option Pricing

What should be the form of the Hamiltonian driving option pricing? Assume that H has the following fairly general form

$$H = a + b \frac{\partial}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \quad (21)$$

Consider for starters the price of a put option. Suppose the strike price $K \rightarrow +\infty$; then the payoff function has the following limit $h(S) = (K - S)_+ \rightarrow K$: constant. Hence,

similar to eq. 16

$$P(t, x) = \langle x | e^{-(T-t)H} | h \rangle \quad (22)$$

$$\rightarrow \langle x | e^{-(T-t)H} | K \rangle \quad (23)$$

$$= e^{-a\tau} K \quad (24)$$

Since $K \rightarrow +\infty$, the option is certain to be exercised, and the holder of the option, in exchange for the stock he holds, is going to be paid an amount K at future time T . Its present-day value, from the principle of no arbitrage, must be the value of K discounted to the present by the risk-free spot interest rate. Hence

$$a = r \quad (25)$$

Consider the call option $C(t, x; K)$ with strike value $K = 0$, and which yields the payoff function $g(x) = (S - K)_+ \rightarrow (S)_+ = S$. What this means is that for the price of nothing, the holder of the call option will be able to get a stock S . Hence it is certain that the holder of the call option will exercise the option, and hence be in possession of a stock at future time T .

What is the discounted value of a stock, held at time T , at present time $t < T$?

The fundamental theorem of finance states that for options to be free from arbitrage opportunities, the evolution driven by the Hamiltonian H must yield a martingale evolution. The risk-free discounted value of a stock is a martingale; in particular we must

have, from eq. 16

$$C(t, x) = \langle x | e^{-(T-t)H} | g \rangle \quad (26)$$

$$\rightarrow \langle x | e^{-(T-t)H} | S \rangle \quad (27)$$

$$= S(x) \text{ due to martingale condition} \quad (28)$$

$$e^{-(T-t)H} | S \rangle = | S \rangle \Rightarrow H | S \rangle = 0 \quad (29)$$

The martingale evolution is expressed by the fact that the Hamiltonian annihilates the underlying security S ; this fact is of far reaching consequences in finance since it holds for more complicated systems like the forward interest rates. The equation $H | S \rangle = 0$ yields

$$b = \frac{\sigma^2}{2} - r \quad (30)$$

and we obtain the famous Black-Scholes Hamiltonian [2]

$$H = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right) \frac{\partial}{\partial x} + r \quad (31)$$

and which provides the following evolution for the option price in terms of remaining time $\tau = T - t$

$$\frac{\partial C(\tau, x)}{\partial \tau} = -\langle x | H | C \rangle \quad (32)$$

$$= \frac{\sigma^2}{2} \frac{\partial^2 C(\tau, x)}{\partial x^2} - \left(\frac{\sigma^2}{2} - r\right) \frac{\partial C(\tau, x)}{\partial x} - r C(\tau, x) \quad (33)$$

The parameter σ is the called the volatility of the stock price, and indicates the degree to which the evolution of the stock price is random.

In terms of the variable $S = e^x$, the Black-Scholes equation for option pricing is given by

$$\frac{\partial C(\tau, x)}{\partial t} = -\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(\tau, x)}{\partial S^2} - r S \frac{\partial C(\tau, x)}{\partial S} + r C(\tau, x) \quad (34)$$

and which is the usual manner in which this equation appears in the literature in finance [1].

5 Pricing Kernel for the Black-Scholes Hamiltonian

To get a flavor of how the formalism of quantum mechanics can address problems in finance, consider the calculation of the pricing kernel. To evaluate the price of the European call option with constant volatility we have the Feynman-Kac formula

$$C(t, x) = \int_{-\infty}^{\infty} dx' \langle x | e^{-(T-t)H_{BS}} | x' \rangle g(x') \quad (35)$$

where recall from eq.(??)

$$H = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right) \frac{\partial}{\partial x} + r \quad (36)$$

We use the Hamiltonian to compute the pricing kernel

$$p(x, x'; \tau) = \langle x | e^{-\tau H} | x' \rangle ; \tau = T - t \quad (37)$$

The first step in determining the pricing kernel is to find the eigenfunctions of H_{BS} . This can be done efficiently by going to the 'momentum' basis in which H_{BS} is diagonal. The Fourier transform of the $|x\rangle$ basis to momentum space, from eq.(??), is given by

$$\begin{aligned} \langle x | x' \rangle = \delta(x - x') &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')} \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x | p \rangle \langle p | x' \rangle \end{aligned}$$

that yields, for momentum space basis $|p\rangle$ the completeness equation

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} |p\rangle \langle p| = \mathcal{I} \quad (38)$$

with the scalar product

$$\langle x|p \rangle = e^{ipx} ; \quad \langle p|x \rangle = e^{-ipx}. \quad (39)$$

From the definition of the Hamiltonian given in eqs.(??) and (??)

$$\begin{aligned} \langle x|H|p \rangle &\equiv H \langle x|p \rangle = H e^{ipx} \\ &= \left\{ \frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r \right\} e^{ipx} \end{aligned} \quad (40)$$

For reference eq.(??) yields ¹

$$\begin{aligned} \langle p|H|x \rangle &= \langle x|H^\dagger|p \rangle^* = [H^\dagger e^{ipx}]^* \\ &= \left\{ \frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r \right\} e^{-ipx} \end{aligned}$$

It can be seen from eq.(40) that functions e^{ipx} are eigenfunctions of H_{BS} , labelled by the 'momentum' index p . Eq.(38) shows that the eigenfunctions of H_{BS} are complete. Hence

$$p(x, \tau; x') = \langle x|e^{-\tau H}|x' \rangle \quad (41)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|e^{-\tau H}|p \rangle \langle p|x' \rangle \\ &= e^{-r\tau} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{1}{2}\tau\sigma^2 p^2} e^{ip(x-x'+\tau(r-\sigma^2/2))} \end{aligned} \quad (42)$$

Performing the Gaussian integration in eq.(42) above gives the pricing kernel for the

¹One might be tempted to consider evaluating the matrix element $\langle p|H|x \rangle$ by directly differentiating on $|x \rangle$; but $\langle p|\partial/\partial x|x \rangle \neq \partial/\partial x \langle p|x \rangle$ and hence this would give an incorrect answer. The operators $\partial/\partial x$ and H_{BS} are defined by their action on the dual co-ordinate basis $\langle x|$ and not on the basis $|x \rangle$; for a Hermitian Hamiltonian this distinction is irrelevant since both procedures give the same answer - and hence this issue is ignored in quantum mechanics - but this is not so for the non-Hermitian case. In fact, it is precisely the non-Hermitian drift term that comes out with the wrong sign if one acts on the basis $|x \rangle$ with H_{BS} .

Black-Scholes equation

$$\begin{aligned}
 p(x, \tau; x') &= \langle x | e^{-\tau H_{BS}} | x' \rangle \\
 &= e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2} \{x-x'+\tau(r-\sigma^2/2)\}^2}
 \end{aligned} \tag{43}$$

A derivation of the Black-Scholes pricing kernel is given in [?] using the method of Laplace transforms².

Eq.(43) given above is the pricing kernel for the Black-Scholes equation.

Recall $x' = \log(S(T))$, $x = \log(S(t))$ and $\tau = T - t$; the stock price evolves randomly from its given value of $S(t)$ at time t to a whole range of possible values $S(T)$ at time T .

Equation (41) states that $\log(S(T))$ has a normal distribution with mean equal to $\log(S(t)) + (r - \sigma^2/2)(T - t)$ and variance of $\sigma^2(T - t)$ as is expected for the Black-Scholes case with constant volatility .

In general for a more complicated (nonlinear) Hamiltonian it is not possible to exactly diagonalize H , and consequently to exactly evaluate the matrix elements of $e^{-\tau H}$. The Feynman path integral is an efficient theoretical tool for analyzing such nonlinear Hamiltonians, and allows for efficient numerical algorithms.

6 Black-Scholes Option Price

Once we have obtained the pricing kernel, the price of an option can be obtained by Gaussian integration. Due to its widespread usage, an explicit derivation is given of the call option price. Let $S(T) = e^x$, $x_0 = \ln S + \tau(r - \frac{\sigma^2}{2})$. The option price, from eqs.(??)

²Recall $x' = \log(S(T))$, $x = \log(S(t))$ and $\tau = T - t$, and the earlier derivation of the pricing kernel given in eq.(??) has been obtained using the martingale condition.

and (??), is given by

$$\begin{aligned}
C &= e^{-r\tau} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi\tau\sigma^2}} (e^x - K)_+ e^{-\frac{1}{2\tau\sigma^2}(x-x_0)^2} \\
&= e^{-r\tau} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi\tau\sigma^2}} (e^{x+x_0} - K)_+ e^{-\frac{1}{2\tau\sigma^2}x^2} \\
&= e^{-r\tau} \int_{\ln K - x_0}^{+\infty} \frac{dx}{\sqrt{2\pi\tau\sigma^2}} (e^{x+x_0} - K) e^{-\frac{1}{2\tau\sigma^2}x^2} \\
&= S \left[\int_{\ln K - x_0}^{+\infty} \frac{dx}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2}[x+\tau\sigma^2]^2} \right] - e^{-r\tau} KN(d_-) \\
&= SN(d_+) - e^{-r\tau} KN(d_-)
\end{aligned} \tag{44}$$

The result obtained above is the famous Black-Scholes formula for option pricing that till today forms the cornerstone for pricing of complex derivative instruments in the financial markets.

7 Acknowledgements

References

- [1] R. Jarrow and S. Turnbull, *Derivative Securities, Second Edition*, South-Western College Publishing (2000).
- [2] B. E. Baaquie, *Quantum Finance*, Cambridge University Press (2004).
- [3] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modeling*, Springer - Verlag **36** (1997).