Hedging LIBOR derivatives in a field theory model of interest rates

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Abstract

We investigate LIBOR-based derivatives using a parsimonious field theory interest rate model capable of instilling imperfect correlation between different maturities. Delta and Gamma hedge parameters are derived for LIBOR caps against fluctuations in underlying forward rates. An empirical illustration of our methodology is conducted to demonstrate the influence of correlation on the hedging of interest rate risk.

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1. Introduction

All financial instruments are subject to the random behavior of underlying variables such as stock prices, interest rates, exchange rates, etc. There are many ways of defining risk as discussed in Bouchaud and Potters [1]. Hedging is the general concept for the procedure of reducing, if not completely eliminating, an investor’s exposure to randomness by constructing a portfolio of correlated instruments. For the interest rate derivatives discussed in this paper, the underlying source of risk is defined by interest rate fluctuations.

The seminal paper of Black Scholes (1973) [2] was the first to recognize that perfectly hedging a derivative enabled one to price the security by no-arbitrage. Specifically, in the absence of market frictions such as short-selling constraints, the ability to hedge a derivative security coincides with one’s ability to replicate its payoffs. The seller of an option assumes the risk of a potential liability at its maturity. In particular, the buyer of a call option is entitled to receive a non-negative payoff from the seller if the stock price is above a certain threshold. Thus, an increase (decrease) in the stock price increases (decreases) the value of a call option on this underlying stock. However, the terminal value of a call option can be replicated by buying stock and borrowing from the money market account (temporary cash loan). In particular, there exists a trading strategy involving the stock and money market account which forms a replicating portfolio that mimics the call option’s value across time. Intuitively, by purchasing a portion of the underlying stock, fluctuations in the replicating portfolio are

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identical to those of the call option. Therefore, if one sells a call option, they can hedge this possible liability by replicating the option’s payoffs to ensure they have the required funds available to pay the buyer. Hence, selling a call option while replicating its payoffs creates a riskless portfolio containing the call option, a certain amount of stock and the money market (cash) account. The critical amount of stock that needs to be purchased and included in the replicating portfolio is referred to as the option’s delta hedge parameter. Similarly, transactions in the bond and futures market can replicate as well as hedge interest rate options. Overall, ascertaining the trading strategy or delta hedge parameter which replicates a derivative enables one to price this security by no-arbitrage as risk preferences become irrelevant once a riskless portfolio is created by hedging. Moreover, the initial cost involved in forming a replicating portfolio that provides identical payoffs as the derivative must equal the price of the derivative by no-arbitrage (law of one price).

We utilize field theory models introduced by Baaquie [3] to instill imperfect correlation between LIBOR maturities as a parsimonious alternative to the existing theory. We derive the corresponding hedge parameters of LIBOR caplets for applications to risk management. We then demonstrate the ease with which our formulation is implemented and the implications of correlation on the hedge parameters.

Hedge parameters that minimize the risk associated with a finite number of random fluctuations in the forward interest rates is provided in Baaquie et al.[4]. Previously, field theory research has focused on applications involving traditional Heath et al.[5] forward interest rates, and on the pricing of LIBOR-based derivatives as is Baaquie [6]. This paper extends the concept of stochastic delta hedging developed in Baaquie [3] to the hedging of LIBOR derivatives. For details regarding the field theory model and the pricing formula of caplets, we refer the interested reader to the paper entitled ‘Empirical investigation of the price of an interest rate caplet for a field theory formula and Black’s formula’ in this issue.

The remainder of this paper begins with a general review of hedging. We then investigate hedge parameters for LIBOR derivatives in Section 2, while Section 3 details their empirical implementation. The conclusion follows in Section 4.

2. Hedging

This section details the implications of our field theory model on hedging LIBOR derivatives. The impact of correlation is examined in the context of the residual variance and the delta hedge parameter for a portfolio. In particular, a stochastic Delta hedging technique is given in Section 2.1.

A portfolio \( P(t) \) composed of a \( \text{Cap}(t, t_s, T) \) and \( N \) LIBOR futures contracts, with the futures chosen to ensure fluctuations in the value of the portfolio are minimized, is studied. This portfolio equals

\[
P(t) = \text{Cap}(t, t_s, T) + \sum_{i=1}^{N} \eta_i(t) F(t, T_i), \tag{1}
\]

where \( \eta_i(t) \) represents the hedge parameter for the \( i \)th futures contract included in the portfolio. These \( \eta_i(t) \) terms ensure movements in the cap and futures contracts “offset” one another to minimize the fluctuations in \( P(t) \). The LIBOR futures and cap prices are denoted by

\[
\mathcal{F}(t, T_i) = V[1 - e^{-L(t, T_i)}], \tag{2}
\]

\[
\text{Cap}(t, t_s, T) = \tilde{V} B(t, T) \int_{-\infty}^{+\infty} \frac{dG}{\sqrt{2\pi q^*_s}} e^{-\frac{1}{2} q^*_s G} \left[ e_{T+t}^{\tilde{X}-q^*_s/2}(X-e^{-G}) \right], \tag{3}
\]

where for the midcurve cap, we have

\[
q^*_s = \frac{1}{t_s - t} \int_t^{t_s} dt' \int_{T_n}^{T_n+t'} dx' \sigma(t', x') D(x, x'; \sigma(t', x')). \tag{4}
\]

This is a more general expression for a cap referred to as the midcurve cap.
From Eq. (1), we have

$$\Pi(t) = \text{Cap}(t, t, T) + V \sum_{i=1}^{N} \eta_i(t)(1 - \ell L(t, T_i)),$$

For the sake of brevity, we suppress the constant term $V \sum_{i=1}^{N} \eta_i$ in the above equation, which is irrelevant for hedging, and change the negative sign before the LIBOR futures to positive as follows:

$$\Pi(t) = \text{Cap}(t, t, T) + V \sum_{i=1}^{N} \eta_i(t)\ell L(t, T_i)$$

$$= \text{Cap}(t, t, T) + V \sum_{i=1}^{N} \eta_i(t)(e^{\int_{T_i}^{T+t} f(t,x)dx} - 1).$$

(5)

In the next two subsections, we discuss two methods for obtaining the $\eta_i(t)$ parameters. We first study stochastic hedging which requires us to specify which forward rate movements are to be hedged against. We then investigate a portfolio’s residual variance, a technique which enables us to control the effectiveness of the hedging procedure. Specifically, instead of only hedging a subset of forward rate movements, residual variance applies to all forward rate fluctuations, including those that increase the portfolio’s value. The material on stochastic hedging constitutes the main results of this paper.

2.1. Stochastic hedging

Stochastic hedging of interest rate derivatives has been introduced by Baaquie [3], where the specific case of hedging treasury bonds is considered in detail. We focus on applying this technique to the hedging of a LIBOR cap. Consider the hedging of a cap against fluctuations in the forward rates $f(t, x)$ which also influence the futures price.

A portfolio $\Pi(t)$ composed of a Cap$(t, t, T)$ and one LIBOR futures contract is studied. As in Eq. (5), we set $N = 1$ to obtain

$$\Pi(t) = \text{Cap}(t, t, T) + V \eta_1(t)(e^{\int_{T}^{T+t} f(t,x)dx} - 1),$$

where the hedging of this portfolio at instant time $t$ is given by

$$\Delta \Pi(t) = \frac{\partial \Pi}{\partial t} \Delta t + \int dx \frac{\delta \Pi}{\delta f(t,x)} \Delta f(t, x) + \frac{1}{2} \int dx \frac{\delta^2 \Pi}{\delta f(t,x)^2}(\Delta f(t, x))^2$$

$$+ \int dx \int dx' \frac{\delta^2 \Pi}{\delta f(t,x)\delta f(t,x')} \Delta f(t, x)\Delta f(t, x') + O(\varepsilon^2)$$

(6)

where $\Delta f(t, x) = f(t + \varepsilon, x) - f(t, x)$. And $\Delta t = \frac{1}{360}$ year, while higher orders of $\varepsilon$ are negligible. Furthermore, the dynamics $d\Pi/dt = \dot{\Pi} = \Delta \Pi/\Delta t$ equal

$$\dot{\Pi}(t) = \frac{\partial \Pi}{\partial t} + \int dx \frac{\delta \Pi}{\delta f(t,x)} \dot{f}(t, x) + \frac{\varepsilon}{2} \int dx \frac{\delta^2 \Pi}{\delta f(t,x)^2}\dot{f}^2(t, x)$$

$$+ \varepsilon \int dx \int dx' \frac{\delta^2 \Pi}{\delta f(t,x)\delta f(t,x')} \dot{f}(t, x)\dot{f}(t, x') + O(\varepsilon).$$

(7)

Since $\langle \dot{f}(t, x)\dot{f}(t, x') \rangle \sim 1/\varepsilon$ as in Baaquie [3], $c\dot{f}(t, x)^2 \sim 0(1)$, the second-order terms are as important as the first-order terms. Normal calculus retains the first-order terms since $\varepsilon$ is infinitesimally small.

We study the delta hedging first. The portfolio is required to be invariant to small changes in the forward rate. Thus, delta hedging this portfolio requires

$$\frac{\delta}{\delta f(t, x)} \Pi(t) = 0.$$
This delta hedge involves a first-order approximation for the change in a portfolio’s value as a result of forward rate fluctuations.

In field theory, for each time $t$, there are infinitely many random variables driving the forward rate term-structure indexed by $x$. Therefore, for any $N$, one can never perfectly delta hedge by satisfying Eq. (8). The best alternative is to delta hedge on average, and this scheme is referred to as stochastic delta hedging as detailed in Baaquie [3]. To implement stochastic delta hedging, one considers the conditional expectation value of the portfolio $\Pi(t)$, conditioned on the occurrence of some specific value of the forward rate $f_h \equiv f(t, x_h)$, namely $E[\Pi(t)|f(t, x_h)]$. Define the conditional probability of a cap and a LIBOR futures by

$$
\hat{C}ap(t, t_s, T; f_h) = E[\hat{C}ap(t, t_s, T)|f_h], \\
\hat{L}(t, T_1; f_h) = E[\hat{L}(t, T_1)|f_h].
$$

From Baaquie [3], we have the conditional probability of a cap given by

$$
\hat{C}ap(t, t_s, T; f_h) = \hat{V} \int_{-\infty}^{\infty} dG(x - e^G)\Phi(G|f_h),
$$

$$
\Phi(G|f_h) = \frac{\int_{-\infty}^{\infty} (dp/2\pi)e^{-(q_0^2/2)^2} \int Df e^{-\int^{T_{1+t}}_T df(t, x) \delta f(t, x_h) - f}e^S}{\int Df \delta f(t, x_h) - f}e^S,
$$

while the conditional probability of a LIBOR futures is

$$
\hat{L}(t, T_1; f_h) = \int_{-\infty}^{\infty} dGe^G\Phi(G|f; t, T_1),
$$

$$
\Phi(G|f; t, T_1) = \frac{\int Df \delta f - \int^{T_{1+t}}_T df(t, x) \delta f(t, x_h) - f}e^S}{\int Df \delta f(t, x_h) - f}e^S.
$$

Stochastic delta hedging is defined by approximating Eq. (8) as

$$
\frac{\partial}{\partial f_h} E[\Pi(t)|f_h] = 0,
$$

for a given forward rate $f_h$ among an infinite number of possible forward rates. Hence, from Eq. (12), stochastic delta hedging yields

$$
\eta_1 = -\frac{\partial \hat{C}ap(t, t_s, T; f_h)}{\partial f_h} / \frac{\partial \hat{L}(t, T_1; f_h)}{\partial f_h}.
$$

Thus, changes in the hedged portfolio $\Pi(t)$ are, on average, sensitive to fluctuations in the forward rate $f(t, x_h)$.

The conditional probability in Eqs. (10) and (11) along with the hedge parameter $\eta_1$ is evaluated explicitly for the field theory description of forward rates in the appendix which also contains the relevant notation. One should notice that nontrivial correlations appear in all the terms. The final result, from Eq. (42), is given by

$$
\eta_1 = \frac{C \cdot \hat{C}ap(t, t_s, T; f_h) - B \cdot \chi \cdot \hat{V} \cdot |XN'(d_+)|/Q + e^{-G_0 + G^2/2}N(d_-) - e^{-G_0 + G^2/2}N'(d_-)/Q}{e^{G_1 + G^2/2} \cdot B_1}.
$$

As a comparison, the HJM limit is also analyzed in the appendix.

Furthermore, one can gamma hedge the same forward rate. The second-order gamma hedge recognizes that large movements in the forward rate may cause the first-order delta approximation to be inaccurate. In particular, if hedging is not performed frequently, the delta hedge parameter can become outdated. However, gamma evaluates changes in the delta hedge parameter as the forward rate term structure evolves over time.

To hedge against the $\partial^2 \Pi(t)/\partial f_h^2$ fluctuations, one needs to form a portfolio with two LIBOR futures contracts that minimizes the change in the value of $E[\Pi(t)|f_h]$ by both delta and gamma hedging. The hedge parameters are solved analytically, with empirical results presented in Section 4.
Suppose a cap needs to be hedged against the fluctuations of two forward rates, namely $f_h$ for $h = 1, 2$. The conditional probabilities for the cap and LIBOR futures, with two forward rates fixed at $f_h$, are

\[
\hat{C}(t, t_{\ast}, T; f_1, f_2) = E[Cap(t, t_{\ast}, T)|f_1, f_2],
\]

\[
\hat{L}(t, T_1; f_1, f_2) = E[L(t, T_1)|f_1, f_2].
\]

A portfolio of two LIBOR futures contracts with different maturities $T_i \neq T$ is defined as

\[
\Pi(t) = Cap(t, t_{\ast}, T) + \sum_{i=1}^{N} \eta_i(t)L(t, T_i),
\]

where the hedging of this portfolio at instant time $t$ is given by

\[
\delta \Pi(t, f_1, f_2) = \frac{\partial \Pi}{\partial t} \delta t + \sum_{i=1}^{2} \frac{\partial \Pi}{\partial f_i} \delta f_i + \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2 \Pi}{\partial f_i^2} \delta^2 f_i + \frac{1}{2} \frac{\partial^2 \Pi}{\partial f_1 \partial f_2} \delta f_1 \delta f_2 + O(\varepsilon^2).
\]

The dynamics $\dot{\Pi} = \delta \Pi / \delta t$ equal

\[
\dot{\Pi}(t, f_1, f_2) = \frac{\partial \Pi}{\partial f_1} \dot{f}_1 + \frac{\partial \Pi}{\partial f_2} \dot{f}_2 + \frac{\partial^2 \Pi}{\partial f_1 \partial f_2} \dot{f}_1 \dot{f}_2 + O(\varepsilon).
\]

The stochastic delta hedging conditions are given by

\[
\frac{\partial}{\partial f_h} E[\Pi(t)|f_1, f_2] = 0 \quad \text{for} \quad h = 1, 2
\]

while stochastic gamma hedging involves

\[
\frac{\partial^2}{\partial f_h^2} E[\Pi(t)|f_1, f_2] = 0 \quad \text{for} \quad h = 1, 2
\]

with cross gamma hedging

\[
\frac{\partial^2}{\partial f_1 \partial f_2} E[\Pi(t)|f_1, f_2] = 0
\]

being unique to this paper. This cross gamma hedging only make sense in field theory models where movements in any specific forward rate can be hedged.

One can solve the above system of $N$ simultaneous equations to determine the $N$ hedge parameters. The volatility of the hedged portfolio is reduced by increasing the number of forward interest rates being hedged.

For this portfolio, we can analytically prove that delta hedge parameters for the two forward rates differ by a prefactor $A_2 / A_{12}$

\[
\frac{\partial}{\partial f_1} E[\Pi(t)|f_1, f_2] = - \frac{A_2}{A_{12}} \frac{\partial}{\partial f_2} E[\Pi(t)|f_1, f_2],
\]

where $A_2$ and $A_{12}$ are defined in Appendix D. Therefore, delta hedging against two forward rates only determines the portfolio’s hedge parameters for one LIBOR futures. Gamma hedging two forward rates is also the same except for a prefactor $A_2 / A_{12}$.

Overall, for hedging against two forward rates we are left with three independent constraints from the above six constraints. In order to study the effect of each set of constraints separately, we form portfolios which include two LIBOR futures, and adopt hedging strategies that involve more than delta hedging to solve for the two hedge parameters. The first strategy implements one delta and one gamma hedge against a single forward rate. Two hedge parameters can also be solved in the context of a one delta hedge and an additional cross gamma hedge.

All of these hedge strategies are evaluated explicitly in the appendix. Intuitively, we expect the portfolio to be hedged more effectively with the inclusion of the cross gamma parameter. Generally speaking, the field
theory framework allows us to form portfolios that hedge against any number of forward rates by including more LIBOR futures contracts. Until now, we obtained the parameter for each choice of the LIBOR futures and forward rates being hedged. Furthermore, we can minimize the following:

$$\sum_{i=1}^{N} |\eta_i|$$  \hspace{1cm} (19)

to find the minimum portfolio. This additional constraint finds the most effective futures contracts, where effectiveness is measured by requiring the smallest number of contracts, hence transactions.

In general, stochastic delta hedging against $N$ forward rates for large $N$ is complicated, and closed-form solutions are difficult to obtain.

### 2.2. Residual variance

Hedging a cap using LIBOR futures contracts can also be accomplished by minimizing the residual variance of the hedged portfolio. It is the instantaneous change in the portfolio value that is stochastic. Therefore, the volatility of this change is computed to ascertain the efficacy of the hedge portfolio.

In terms of notation, $C(t, T)$ equal $C(t, t_s, T)$ in the previous subsection where $t_s = T$ and $T = t_s + \delta$ for $\delta$ being 3 months. The variance of the portfolio fluctuations, $\text{Var}[dI(t)/dt]$, equals

$$\text{Var}\left[\frac{d\text{Cap}(t, T)}{dt}\right] + \text{Var}\left[\sum_{i=1}^{N} \Delta_i \frac{dL(t, T_i)}{dt}\right]$$

$$+ \sum_{i=1}^{N} \Delta_i \text{Var}\left[\left\langle \frac{d\text{Cap}(t, T)}{dt}, \frac{dL(t, T_i)}{dt} \right\rangle - \left\langle \frac{d\text{Cap}(t, T)}{dt} \right\rangle \left\langle \frac{dL(t, T_i)}{dt} \right\rangle\right].$$  \hspace{1cm} (20)

The detailed calculation for determining the hedge parameters and portfolio variance is carried out in the appendix. As in Baaquie et al. [4], the following notation is introduced for simplicity:

$$K_i = \chi \hat{L}(t, T_i) \int_{t}^{T_i+\delta} dx \int_{T_j}^{T_i+\delta} dx' \sigma(t, x)\sigma(t, x')D(x, x'; t),$$

$$M_{ij} = \hat{L}(t, T_i)\hat{L}(t, T_j) \int_{t}^{T_i+\delta} dx \int_{T_j}^{T_i+\delta} dx' \sigma(t, x)\sigma(t, x')D(x, x'; t).$$  \hspace{1cm} (21)

Eq. (21) allows the residual variance in Eq. (37) to be succinctly expressed as

$$\chi^2 \int_{t}^{T_i+\delta} dx \int_{t}^{T_i+\delta} dx' \sigma(t, x)\sigma(t, x')D(x, x'; t) + 2 \sum_{i=1}^{N} \Delta_i K_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \Delta_i \Delta_j M_{ij}$$  \hspace{1cm} (22)

which contains covariance terms. When at-the-money, the value of $\chi$ below facilitates our empirical estimation of the model in Section 4

$$\chi = -VB(t, T) \int_{-\infty}^{+\infty} \frac{dG}{\sqrt{2\pi q^2}} q^2 \left( G - \int_{t}^{T_i+\delta} dx f(t, x) - \frac{q^2}{2} \right)$$

$$\times \left\{ e^{(-1/2)q^2 G - q^2 f(t,x) - q^2/2} (X - e^{-G})_+ \right\}$$

$$= VB(t, T) \left\{ \frac{1}{\sqrt{2\pi q^2}} e^{-1/2d^2_+} + \left( \frac{1 + \ell K}{1 + \ell L} \right) \left[ -\frac{1}{\sqrt{2\pi q^2}} e^{-1/2d^2_-} + N(d_-) \right] \right\},$$  \hspace{1cm} (23)

where $d_{\pm} = (\ln X/F \pm q^2/2)/q$. The value of $\chi$ for an at-the-money options yields $d_{\pm} = \pm q/2$ which implies

$$\chi(t, T)|_{\text{at-the-money}} = VB(t, T) N(d_-).$$  \hspace{1cm} (24)
Observe that the residual variance depends on the correlation between forward rates described by the propagator. Ultimately, the effectiveness of the hedge portfolio is an empirical question since perfect hedging is not possible. This empirical question is addressed in Section 3 when the propagator is calibrated to market data.

Hedge parameters $A_i$ that minimize the residual variance in Eq. (22) are

$$A_i = -\sum_{j=1}^{N} K_j M^{-1}_{ij}. \quad (25)$$

These parameters represent the optimal amounts of the futures contracts to include in the hedge portfolio.

Eq. (25) is proved by differentiating Eq. (22) with respect to $A_i$ and subsequently solving for its value. The variance of the hedged portfolio in Eq. (26) is proved by substituting the result of Eq. (25) into Eq. (22)

$$V_R = \chi^2 \int_T^{T+\ell} dx \int_T^{T+\ell} dx' \sigma(t, x) \sigma(t, x') D(x, x'; t) - \sum_{i=1}^{N} \sum_{j=1}^{N} K_i M^{-1}_{ij} K_j \quad (26)$$

which declines monotonically as $N$ increases.

The residual variance in Eq. (26) enables the effectiveness of the hedge portfolio to be evaluated. Therefore, Eq. (26) is the basis for studying the impact of including different LIBOR futures contracts in the hedge portfolio. For $N = 1$, a single maturity $T_i$ is evaluated, and the residual variance in Eq. (26) reduces to

$$\chi^2 \int_T^{T+\ell} dx \int_T^{T+\ell} dx' \sigma(t, x) \sigma(t, x') D(x, x'; t)$$

$$- \left( \left( \int_T^{T+\ell} dx \int_T^{T+\ell} dx' \sigma(t, x) \sigma(t, x') D(x, x'; t) \right)^2 \right). \quad (27)$$

The second term in Eq. (27) represents the reduction in variance attributable to the hedge portfolio. To obtain the HJM limit, the propagator is constrained to equal one, reducing the residual variance $V_R$ in Eq. (27)

$$\chi^2 \left[ \left( \int_T^{T+\ell} dx \sigma(t, x) \right)^2 - \left( \int_T^{T+\ell} dx \sigma(t, x) \sigma(t, x') \right)^2 \right] \quad (28)$$

to zero. This HJM limit is consistent with our intuition that the residual variance is identical to zero for any LIBOR maturity since all forward rates are perfectly correlated. This result is also shown empirically in Section 3. However, results from hedging with two LIBOR futures contracts in HJM model are not presented since one degree of freedom cannot be hedged with two instruments. Indeed, in this circumstance, $M^{-1}$ is singular.

3. Empirical implementation

This section illustrates the implementation of our field theory model and provides preliminary results for the impact of correlation on the hedge parameters. The correlation parameter for the propagator of LIBOR rates is estimated from historical data on LIBOR futures and at-the-money options. We calibrate the term structure of the volatility, $\sigma(0)$, (see [7,8]) and the propagator with the parameters $\lambda$ and $\mu$ as in Baaquie and Bouchaud [9]. All the empirical results showed below are calculated from the derivation expressed in this paper.

3.1. Empirical results on stochastic hedging

Stochastic hedging mitigates the risk of fluctuations in specified forward rates. The focus of this section is on the stochastic hedge parameters $\eta_i$, with the best strategy chosen to ensure the LIBOR futures portfolio involves the smallest possible long and short positions since $\sum_{i=1}^{N} |\eta_i|$ is minimized.
3.1.1. Hedging in field theory models compared to HJM

The comparison is carried out in the simplest portfolio where one forward rate is hedged by one LIBOR futures, with a detailed empirical study in Section 3.1.2. As an illustration, Fig. 1 plots the hedge parameter \( Z_1 \) in our field theory model against the LIBOR futures maturity \( T_1 \) and the forward rate maturity \( x_h \) involving \( \Pi(t) = Cap(t, 1, 4) + Z_1(t)F(t, T_1) \).

Fig. 1. Hedge parameter \( \eta_1 \) for stochastic delta hedging of \( Cap(t, 1, 4) \) using LIBOR futures maturity \( T_1 \) and forward rate maturity \( x_h \) involving \( \Pi(t) = Cap(t, 1, 4) + \eta_1(t)F(t, T_1) \).

Fig. 2. Hedge parameter \( \eta_1 \) for stochastic hedging of \( Cap(t, 1, 4) \) using LIBOR futures maturity \( T_1 \) and forward rate maturity \( x_h \) in the HJM limit of \( D = 1 \) (forward rates perfectly correlated) involving \( \Pi(t) = Cap(t, 1, 4) + \eta_1(t)F(t, T_1) \).

3.1.1. Hedging in field theory models compared to HJM

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model and a single-factor HJM model, we plot the identical hedge portfolio as above when \( D = 1 \), which has been shown to be the HJM limit of field theory models. From Fig. 2, for the HJM limit, the hedge parameter \( \eta_1 \) is invariant to the forward rate maturity \( x_h \), which is expected since all forward rates \( f(t, x_h) \) are perfectly correlated in a single-factor HJM model. Therefore, it makes no difference which of the forward rates is being hedged.

3.1.2. Hedging against one forward rate with one LIBOR futures

We first study a portfolio with one LIBOR futures and one cap to hedge against a single term structure movement. The portfolio is given by

\[
\Pi(t) = Cap(t, t_1, T) + \eta_1(t)F(t, T_1),
\]

where the hedging is done by stochastic delta hedging \( (\partial/\partial f_t)E[\Pi(t)|f_t] = 0 \) on forward rate \( f(t, x_h) \).

\[
\text{Fig. 3. Hedge parameter } \eta_1 \text{ for stochastic hedging of } Cap(t, 1, 4) \text{ for forward maturity } x_h \text{ of forward rate } f(t, x_h), \text{ with fixed LIBOR futures contract maturity } T_1, \text{ involving } \Pi(t) = Cap(t, 1, 4) + \eta_1(t)\mathcal{F}(t, T_1).
\]

\[
\text{Fig. 4. Hedge parameter } \eta_1 \text{ for stochastic hedging of } Cap(t, 1, 4) \text{ for LIBOR futures maturity } T_1 \text{ when hedging against } f(t, t + \delta) \text{ with } \delta = \frac{1}{12}, \text{ involving } \Pi(t) = Cap(t, 1, 4) + \eta_1(t)\mathcal{F}(t, T_1).
\]
Hedge parameters $\eta_1$, for different LIBOR futures maturities $T_1$, and the forward rate maturity $x_h$, are shown in Fig. 1. This figure describes the selection of the LIBOR futures in the minimum portfolio that requires the fewest number of long and short positions.

Fig. 3 shows how the hedge parameters depend on $x_h$ for a fixed $T$. Two limits $T_1 = \delta = \frac{1}{2}$ (3 months) and $T_1 = 16\delta$ are chosen. We find that $x_h = \delta$ is always the most important forward rate to hedge against. Another graph describing the parameter dependence on $T_1$ is given in Fig. 4 with $x_h = \delta$. The minimum of hedge parameter $Z_1$ at $x_h' = 5$ years reflects the maximum of $s(t, x)$ around the same future time. For greater generality, we also hedge $C(t, t, T)$ for different $t$ and $T$ values, and find that although the value of the parameter changes slightly, the shape of the parameter surface is almost identical.

3.1.3. Hedging against one forward rate with two LIBOR futures

In Fig. 5, we investigate hedging one forward rate with two LIBOR futures by employing both delta and gamma hedging. The portfolio is given by

$$P(t) = Cap(t, t + \delta) + \sum_{i=1}^{2} \eta_i(t) \mathcal{F}(t, T_i),$$

where stochastic delta hedging $\left(\partial/\partial f_1\right)E[\Pi(t)|f_1] = 0$ and stochastic gamma hedging $\left(\partial^2/\partial f_1^2\right)E[\Pi(t)|f_1] = 0$ are employed.

From the previous case, we can hedge against $f(t, \delta)$ in order to obtain a minimum portfolio involving the least amount of short and long positions. The diagonal reports that two LIBOR futures with the same maturity reduces to delta hedging with one LIBOR futures. The data from which Fig. 5 is plotted illustrates that selling 38 contracts of $L(t, t + 6\delta)$ and buying 71 $L(t, t + \delta)$ contracts identifies the minimum portfolio. More explicitly, the variables in the portfolio are given as

<table>
<thead>
<tr>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$x_{h1}$</th>
<th>$\eta_1$</th>
<th>$\eta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5 year</td>
<td>0.25 year</td>
<td>0.25 year</td>
<td>-38</td>
<td>71</td>
</tr>
</tbody>
</table>
3.1.4. Hedging against two forward rates with two LIBOR futures

In addition, we consider hedging fluctuations in two forward rates. Specifically, we study a portfolio comprised of two LIBOR futures and one caplet

$$P(t) = \text{Cap}(t, 1, 4) + \sum_{i=1}^{2} \eta_i(t) \mathcal{F}(t, T_i)$$

where the parameters $\eta_i$ are fixed by delta hedging ($\frac{\partial}{\partial f_1} E[I(t)|f_1, f_2] = 0$) and cross gamma hedging ($\frac{\partial^2}{\partial f_1 \partial f_2} E[I(t)|f_1, f_2] = 0$).

The result is displayed in Fig. 6 where we hedge against two short maturity forward rates, such as $f(t, \delta)$ and $f(t, 2\delta)$. Again the data from which Fig. 6 is plotted illustrates that buying 45 contracts of $L(t, t + 15\delta)$ and selling 25 $L(t, t + 3\delta)$ contracts forms the minimum portfolio. More explicitly, the variables in the portfolio are given as

<table>
<thead>
<tr>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$x_{h1}$</th>
<th>$x_{h2}$</th>
<th>$\eta_1$</th>
<th>$\eta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.75 year</td>
<td>0.75 year</td>
<td>0.25 year</td>
<td>0.5 year</td>
<td>45</td>
<td>-25</td>
</tr>
</tbody>
</table>

Figs. 5 and 6 result from summing the absolute values of the hedge parameters (as in Eq. (19)) which depend on the maturities of the LIBOR futures $T_i$. The corresponding empirical results are consistent with our earlier discussion.\(^2\)

3.2. Empirical results on residual variance

The reduction in variance achievable by hedging a cap with LIBOR futures is the focus of this section. The portfolio

$$II(t) = \text{Cap}(t, T) + \sum_{i=1}^{N} \Delta_i(t) \mathcal{F}(t, T_i)$$

\(^2\)If we choose the hedge portfolio by minimizing $\sum_{i=1}^{N} \eta_i$, we find that the minimum portfolio requires 1500 contracts (long their short maturity and short their long maturity counterparts).
is considered with $Var[dII(t)/dt]$ being minimized. The residual variance for hedging a 1 year and 4 year cap with a LIBOR futures is shown in Fig. 7, along with its HJM counterpart. Observe that the residual variance drops to exactly zero when the same maturity LIBOR futures is used to hedge the cap.

By considering the changes of residual variance with respect of parameters $\lambda$ and $\mu$, we find the neighboring points create no disparities, at least one cannot tell which offers the better hedge. An explanation of this effect is that forward rates with similar maturities are strongly correlated. Furthermore, the HJM residual variance for both hedging a 1 year and 4 year cap are identical to the residual variance $0$ axis. This is consistent with our analytical result in Eq. (28).

The residual variance for hedging a 4 year cap with two LIBOR futures is provided in Fig. 8. It is interesting to note that hedging with two instruments, even with similar maturities, entails a significant decrease in
residual variance compared to hedging with one futures. This is illustrated in Fig. 8 where \( \theta = 0' \) represents hedging with one LIBOR futures. The residual variance in this situation is higher than the nearby points, and increases in a discontinuous manner.

4. Conclusion

LIBOR-based caps and floors are important financial instruments for managing interest rate risk. However, the multiple payoffs underlying these contracts complicates their pricing as the LIBOR term structure dynamics are not perfectly correlated. A field theory model which allows for imperfect correlation between every LIBOR maturity overcomes this difficulty while maintaining model parsimony.

Furthermore, hedge parameters for the field theory model are provided for risk management applications. Although the field theory model implies an incomplete market since hedging cannot be conducted with an infinite number of interest rate-dependent securities in practice, the correlation structure between LIBOR rates is exploited to minimize risk. An empirical illustration demonstrates the implementation of our model.

Acknowledgment

The data in our empirical tests was generously provided by Jean-Philippe Bouchaud of Science and Finance, and consists of daily closing prices for quarterly Eurodollar futures contracts as described in Bouchaud et al. [7] as well as Bouchaud and Matacz [8].

Appendix A. Residual variance

First, consider the variance of a cap in the field theory model. Define the delta of the cap, \( \delta \text{Cap}(t, T)/\delta \int_{T}^{T+\ell} f(t, x) \, dx \), as \( \chi \)

\[
\chi = - VB(t, T) \int_{-\infty}^{+\infty} \frac{dG}{\sqrt{2\pi q^2 g}} \left( G - \int_{T}^{T+\ell} dx f(t, x) - \frac{q^2}{2} \right) \times \left\{ e^{-(1/2q^2)(g-\int_{T}^{T+\ell} dx f(t, x)-q^2/2)^2} (X - e^{-g})_+ \right\}.
\]

The result in Eq. (3) for the cap price implies that

\[
\frac{d \text{Cap}(t, T)}{dt} = - VB(t, T) \int_{-\infty}^{+\infty} \frac{dG}{\sqrt{2\pi q^2 g}} \left( G - \int_{T}^{T+\ell} dx f(t, x) - \frac{q^2}{2} \right) \int_{T}^{T+\ell} \frac{\partial f(t, x)}{\partial t} \, dx
\]

\[
= \chi \left( \int_{T}^{T+\ell} \frac{\partial f(t, x)}{\partial t} \, dx \right)
\]

\[
= \chi \left( \int_{T}^{T+\ell} \, dx \, \sigma(t, x) + \int_{T}^{T+\ell} \, dx \, \sigma(t, x) \, A(t, x) \right)
\]

with \( E[d \text{Cap}(t, T)/dt] = (\int_{T}^{T+\ell} \, dx \, \sigma(t, x)) \, dt \) since \( E[A(t, x)] = 0 \). Therefore, the resulting variance equals

\[
\frac{d \text{Cap}(t, T)}{E} - E \left[ \frac{d \text{Cap}(t, T)}{E} \right] = \chi \int_{T}^{T+\ell} \, dx \, \sigma(t, x) \, A(t, x).
\]

With \( \delta(\cdot) = 1/\varepsilon \) representing a delta function, squaring this expression and invoking the property that \( E[A(t, x) A(t, x')] = \delta(0) D(x, x'; t) = D(x, x'; t)/dt \) results in the instantaneous cap price variance being

\[
\text{Var} \left[ \frac{d \text{Cap}(t, T)}{E} \right] = \frac{1}{\varepsilon} \chi^2 \int_{T}^{T+\ell} \, dx \int_{T}^{T+\ell} \, dx' \sigma(t, x) D(x, x'; t) \sigma(t, x').
\]
The quantity \( \varepsilon \) signifies a small step forward in time. The underlying intuition is that we are converting a portfolio of futures contracts to one involving another function of LIBOR rates. Then, the instantaneous variance of a LIBOR portfolio is considered. For a LIBOR portfolio, \( \hat{H}(t) = V \int_{t_{i}}^{T} \hat{D}(t, T_{i}) \), the following result holds:

\[
\frac{d\hat{H}(t)}{dt} = E \left[ \frac{d\hat{H}(t)}{dt} \right] = \sum_{i=1}^{N} A_{i} \hat{L}(t, T_{i}) \int_{T_{i}}^{T} dx \sigma(t, x) A(t, x),
\]

(35)

where \( \hat{L}(t, T_{i}) = \exp \left[ T_{i}^{T} f(x) dx \right] = V f(t, T_{i}, T_{i} + \varepsilon) \) and

\[
Var \left[ \frac{d\hat{H}(t)}{dt} \right] = \frac{1}{\varepsilon} \sum_{i=1}^{N} \sum_{j=1}^{N} A_{i} A_{j} \int_{T_{i}}^{T} dx \int_{T_{j}}^{T} dx \sigma(t, x) \sigma(t, x') D(x, x'; t).
\]

(36)

The (residual) variance of the hedged portfolio

\[
\Pi(t) = \text{Cap}(t, T) + \sum_{i=1}^{N} A_{i} \tilde{\mathcal{F}}(t, T_{i})
\]

is then computed in a straightforward manner. Eq. (36) implies the hedged portfolio’s variance equals

\[
\chi^{2} \int_{T}^{T+\varepsilon} dx \int_{T}^{T+\varepsilon} dx' \sigma(t, x) \sigma(t, x') D(x, x'; t)
\]

\[
+ 2\chi \sum_{i=1}^{N} A_{i} \hat{L}(t, T_{i}) \int_{T}^{T+\varepsilon} dx \int_{T_{i}}^{T+\varepsilon} dx' \sigma(t, x) \sigma(t, x') D(x, x'; t)
\]

\[
+ \sum_{i=1}^{N} \sum_{j=1}^{N} A_{i} A_{j} \hat{L}(t, T_{i}) \hat{L}(t, T_{j}) \int_{T_{i}}^{T+\varepsilon} dx \int_{T_{j}}^{T+\varepsilon} dx' \sigma(t, x) \sigma(t, x') D(x, x'; t).
\]

(37)

**Appendix B. Conditional probability of hedging one forward rate**

Using the results of the Gaussian models in Baaquie [3], after a straightforward but tedious calculation, the following is derived from Eqs. (10) and (11):

\[
\Psi(G|f_{h}) = \frac{1}{\sqrt{2\pi Q}} \exp \left[ -\frac{1}{2Q}(G - G_{0})^{2} \right],
\]

(38)

\[
\Phi(G|f_{h}, T_{n}) = \frac{1}{\sqrt{2\pi Q_{1}^{2}}} \exp \left[ -\frac{1}{2Q_{1}}(G - G_{1})^{2} \right].
\]

(39)

The notations are shown as follows:

\[
X = \frac{1}{1 + \varepsilon k}, \quad \tilde{\mathcal{V}} = (1 + \varepsilon k) V,
\]

\[
\chi = \exp \left\{ - \int_{T_{h}}^{T_{n}} dx f(t_{0}, x) - \int_{t_{h}}^{t} z(t, x) + \frac{1}{2} E + \frac{C}{A} \left( f(t_{0}, x_{h}) + \int_{t_{h}}^{t} dt z(t, x_{h}) - f - \frac{C}{2} \right) \right\},
\]

\[
d_{+} = (\ln x + G_{0})/Q, \quad d_{-} = (\ln x + G_{0} - Q^{2})/Q,
\]

\[
G_{0} = \int_{T_{n}}^{T_{n}+\varepsilon} dx f(t_{0}, x) - F - \frac{B}{A} \left( f(t_{0}, x_{h}) - C - f + \int_{t_{h}}^{t} dt z(t, x_{h}) + q^{2}/2 \right).
\]
\[ Q^2 = q^2 - \frac{B_1^2}{A}, \]
\[ G_1 = \int_{T_n}^{T_n+\ell} dx f(t_0, x) + \int_{M_2} dz(t, x) - \frac{B_1}{A} \left( f(t_0, x_h) - \int_{t_0}^{t_h} dt z(t, x_h) - f \right), \]
\[ Q_1^2 = D - \frac{B_1^2}{A}, \]
\[ A = \int_{t_0}^{t_h} dt \sigma(t, x_h)^2 D(t, x_h, x_h; T_{FR}), \]
\[ B = \int_{M_2} \sigma(t, x_h) D(t, x_h, x; T_{FR}) \sigma(t, x), \]
\[ B_1 = \int_{M_1} \sigma(t, x_h) D(t, x_h, x; T_{FR}) \sigma(t, x), \]
\[ C = \int_{M_1} \sigma(t, x_h) D(t, x_h, x; T_{FR}) \sigma(t, x), \]
\[ D = \int_{\mathbb{H}_1} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'), \]
\[ q^2 = \int_{\mathbb{H}_2+\mathbb{H}_4} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'), \]
\[ E = \int_{\mathbb{H}_1} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'), \]
\[ F = \int_{t_0}^{t_h} dt \int_{t_h}^{T_n} dx \int_{T_n}^{T_n+\ell} dx' \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'). \]

The domain of integration is given in Figs. 9 and 10. It can be seen that the unconditional probability distribution for the cap and LIBOR futures yields volatilities \( q^2 \) and \( D \), respectively. Hence the conditional expectation reduces the volatility of cap by \( B_1^2 / A \), and by \( B_1^2 / A \) for the LIBOR futures. This result is expected since the constraint imposed by the requirement of a conditional probability reduces the allowed fluctuations of the instruments.

---

**Fig. 9.** Domain of integration \( M_1, M_2 \) and integration cube \( \mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_4 \) where the \( x' \)-axis has the same limit as its corresponding \( x \)-axis.
It could be the case that there is a special maturity time $x_h$ which causes the largest reduction in conditional variance. The answer is found by minimizing the conditional variance

$$\nonumber \begin{align*}
\text{Cap} (t_h, x_N; f_h) &= \hat{\gamma} \hat{V} (x N (d_+) - e^{-G_0 + Q^2/2} N (d_-)), \\
\hat{L} (t_h, x_N; f_h) &= e^{G_1 + Q^2/2}.
\end{align*}$$

Recall the hedging parameter is given by Eq. (13). Using Eq. (41) and setting $t_0 = t$, $t_h = t + \varepsilon$, we get an (instantaneous) stochastic delta hedge parameter $\eta (t)$ equal to

$$\nonumber \begin{align*}
C \cdot \text{Cap} (t, x_N; f_h) - B \cdot \hat{\gamma} \cdot \hat{V} \cdot [x N (d_+) / Q + e^{-G_0 + Q^2/2} N (d_-) - e^{-G_0 + Q^2/2} N (d_-) / Q] \\
&= e^{G_1 + Q^2/2} \cdot B_i.
\end{align*}$$

### Appendix C. HJM limit of hedging function

The HJM-limit of the hedging functions is analyzed for the specific exponential function considered by Jarrow and Turnbull [10] 

$$\nonumber \begin{align*}
\sigma_{hjm} (t, x) &= \sigma_0 e^{\beta (x-t)},
\end{align*}$$

which sets the propagator $D (t, x, x'; T_{FR})$ equal to one. It can be shown that

$$\nonumber \begin{align*}
A &= \frac{\sigma_0^2}{2 \beta} e^{-2 \beta y} (e^{2 \beta \beta_0} - e^{2 \beta_0}), \\
B &= \frac{\sigma_0^2}{2 \beta^2} e^{-\beta y} (e^{-\beta T_N} - e^{-\beta T_n + \varepsilon}) (e^{2 \beta \beta_0} - e^{2 \beta_0}), \\
B_1 &= \frac{\sigma_0^2}{2 \beta^2} e^{-\beta y} (e^{-\beta T_n} - e^{-\beta T_n + \varepsilon}) (e^{2 \beta \beta_0} - e^{2 \beta_0}), \\
C &= \frac{\sigma_0^2}{2 \beta^2} e^{-\beta y} (e^{-\beta T_N} - e^{-\beta T_n}) (e^{2 \beta \beta_0} - e^{2 \beta_0}), \\
D &= \frac{\sigma_0^2}{2 \beta^2} (e^{-\beta T_n + \varepsilon} - e^{-\beta T_n})^2 (e^{2 \beta \beta_0} - e^{2 \beta_0}), \\
E &= \frac{\sigma_0^2}{2 \beta^2} (e^{-\beta T_n} - e^{-\beta \beta_0})^2 (e^{2 \beta \beta_0} - e^{2 \beta_0}).
\end{align*}$$
\[ F = \frac{\sigma_0^2}{2b^3} (e^{-\beta t_n+\ell} - e^{-\beta t_n}) (e^{-\beta t_n} - e^{-\beta t_0})(e^{2\beta t_h} - e^{2\beta t_0}). \]

The exponential volatility function given in Eq. (43) has the remarkable property, similar to the case found for the hedging of treasury bonds in Baaquie [3], that

\[ Q_1^2(hjm) = D_{hjm} - \frac{B_{hjm}^2}{A_{hjm}} = 0. \quad (44) \]

Hence, the conditional probability for the LIBOR futures is deterministic. Indeed, once the forward rate \( f_h \) is fixed, the following identity is valid:

\[ \tilde{L}_{hjm}(t_h, T_{n1}; f_h) \equiv L(t_h, T_{n1}). \quad (45) \]

In other words, for the volatility function in Eq. (43), the LIBOR futures for the HJM model is exactly determined by one of the forward rates.

However, the conditional probability for the cap is not deterministic since the volatility from \( t_h \) to \( t_n \), before the cap’s expiration, is not compensated for by fixing the forward rate.

**Appendix D. Conditional probability of hedging two forward rates**

When hedging against two forward rates, Eqs. (10) and (11) imply we have the conditional probability of a cap given by

\[ \Psi(G|f_1, f_2) = \int_{-\infty}^{\infty} (dp/2\pi) e^{(G - q_j^2/2)p^2} e^{ip(G - q_j^2/2)} \int Df e^{-\int_{t_h}^{T_n} f(t_n, x) dx} e^{ip\int_{t_h}^{T_n} f(t_n, x) dx} \Delta_i^2 \delta(f(t_h, x_i) - f_j)e^S, \]

and the conditional probability of LIBOR being

\[ \Phi(G|f_1, f_2, T_{nj}) = \int Df \delta(G - \int_{t_h}^{T_n} f(t_h, x) dx) \Delta_i^2 \delta(f(t_h, x_i) - f_j)e^S, \]

which yields

\[ \Psi(G|f_1, f_2) = \frac{\chi}{\sqrt{2\pi Q}} \exp \left\{ -\frac{1}{2Q^2} (G - G_0)^2 \right\}, \quad (48) \]

\[ \Phi(G|f_1, f_2, T_{nj}) = \frac{1}{\sqrt{2\pi \tilde{Q}^2}} \exp \left\{ -\frac{1}{2\tilde{Q}^2} (G - \tilde{G}_j)^2 \right\}, \quad j = 1, 2 \quad (49) \]

under the following notation:

\[ \chi = \frac{1}{1 + \ell k}, \quad \tilde{V} = (1 + \ell k) V, \]

\[ \tilde{Q} = \exp \left\{ -\int_{t_h}^{T_n} dx f(t_0, x) - \int x(t, x) - \frac{1}{2} E + \frac{C_{12}}{A_{12}} \left( R_{12} - \frac{C_{12}}{2} \right) \right\}, \]

\[ d_+ = (\ln x + G_0)/Q, \quad d_- = (\ln x + G_0 - Q^2)/Q, \]
\[ G_0 = \int_{T_n}^{T_{n+\ell}} dx f(t_0, x) - F - \frac{B_{12}}{A_{12}} (R_{12} - C_{12}) + \frac{q^2}{2}, \]
\[ Q^2 = q^2 - \frac{B_{12}^2}{A_{12}}, \]
\[ \tilde{G}_j = \int_{T_{nj}}^{T_{nj+\ell}} dx f(t_0, x) + \int M_j x(t, x) - \frac{\tilde{B}_{12j}}{A_{12}} R_{12}, \quad j = 1, 2, \]
\[ \tilde{Q}_{ij}^2 = D_{ij} - \frac{B_{12j}^2}{A_{12}}, \quad j = 1, 2, \]
\[ R_i = f(t_0, x_i) + \int_{t_0}^{t_h} dt x(t, x_i) - f_i, \quad i = 1, 2, \]
\[ R_{12} = R_1 - \frac{A_{12}}{A_2} R_2, \]
\[ A_i = \int_{M_2} \sigma(t, x_i)^2 D(t, x_i, x_i; T_{FR}), \quad i = 1, 2, \]
\[ A_{12} = \int_{M_2} \sigma(t, x_1) D(t, x_1, x_2; T_{FR}) \sigma(t, x_2), \]
\[ A_{12} = A_1 - \frac{A_{12}}{A_2} A_2, \]
\[ B_i = \int_{M_1} \sigma(t, x_i) D(t, x_i, x; T_{FR}) \sigma(t, x), \quad i = 1, 2, \]
\[ B_{12} = B_1 - \frac{A_{12}}{A_2} B_2, \]
\[ \tilde{B}_{ij} = \int_M \sigma(t, x_i) D(t, x_i, x; T_{FR}) \sigma(t, x), \quad i = 1, 2, \quad j = 1, 2, \]
\[ \tilde{B}_{12j} = \tilde{B}_{1j} - \frac{A_{12}}{A_2} \tilde{B}_{2j}, \quad j = 1, 2, \ldots, 5, \]
\[ C_i = \int_{M_1} \sigma(t, x_i) D(t, x_i, x; T_{FR}) \sigma(t, x), \quad i = 1, 2, \]
\[ C_{12} = C_1 - \frac{A_{12}}{A_2} C_2, \]
\[ D_j = \int_{\hat{\beta}_j} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'), \quad j = 1, 2, \]
\[ q^2 = \int_{\hat{\beta}_2 + \hat{\beta}_4} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'), \]
\[ E = \int_{\hat{\beta}_1} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'), \]
\[ F = \int_{t_0}^{t_h} dt \int_{t_0}^{t_h} dx \int_{t_n}^{T_n} dx' \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x'). \]

The domain of integration is given in Figs. 9 and 10.

Furthermore, an \( N \)-fold constraint on the instruments would further reduce the variance of the instruments.

\[ \tilde{C} \psi_1 (t_h, t_+, T_{n}, f_1, f_2) = \chi \tilde{V} (x N(d_+) - e^{-\tilde{G}_{0+} \tilde{Q}_1^2/2} N(d_-)), \]
\[ \tilde{L}(t_h, T_{nj}; f_1, f_2) = \epsilon \tilde{G}_{ij} + \tilde{Q}_{ij}^2 \frac{1}{2}. \]
References


