

Empirical investigation of a field theory formula and Black's formula for the price of an interest-rate caplet

Belal E. Baaquie*, Cui Liang

Department of Physics, National University of Singapore, Kent Ridge, Singapore 117542, Singapore

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Abstract

The industry standard for pricing an interest-rate caplet is Black's formula. Another distinct price of the same caplet can be derived using a quantum field theory model of the forward interest rates. An empirical study is carried out to compare the two caplet pricing formulae. Historical volatility and correlation of forward interest rates are used to generate the field theory caplet price; another approach is to fit a parametric formula for the effective volatility using market caplet price. The study shows that the field theory model generates the price of a caplet and cap fairly accurately. Black's formula for a caplet is compared with field theory pricing formula. It is seen that the field theory formula for caplet price has many advantages over Black's formula.

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1. Introduction

London interbank overnight rates (Libor), which are interest rates for Eurodollar deposits, can have derivatives written on them such as caps and floors; these instruments are important interest-rate derivatives and have many applications in the financial markets.¹ Interest-rate contracts cover many years and can involve a sequence of quarterly payments ranging from 1 to 10 years. Consequently, pricing and hedging such derivatives requires the modelling of Libor over a long interval of time.

In an economy where Libor rates are perfectly correlated across different maturities, a single volatility function is sufficient. However, non-parallel movements in the Libor term structure introduce an important complication. To reduce the number of necessary inputs, volatility parameters within certain time intervals are often assumed to be constant and lead to many inaccuracies. Furthermore, longer maturity options still require a large number of volatility parameters even after such aggregation.

Liquid interest-rate option like cap and floor have been embedded with all available information in the price, thus the main challenge for market participants is to extract information from these options and use

*Corresponding author.

E-mail address: phybeb@nus.edu.sg (B.E. Baaquie).

¹Libor and caplets are briefly discussed in Appendix A.

these information and a no-arbitrage model to price other exotic options. The underlying Libor rates are common for these options, consequently one may need to extract information purely depending on the Libor rates.

In Black's formula for caplets the implied volatility σ_B for a caplet is quoted in the market and the market prices are computed from this by Black's formula, which is similar to the Black–Scholes option formula [1]. Caplets at different maturities and different strike prices are quoted with different σ_B which means σ_B is not purely dependent on Libor and cannot be used to price other Libor options. Also, the Black's formula in effect has no predictive power but instead is used as a non-linear transformation from caplet volatilities to prices: implied volatility is simply another way of quoting the price of the caplet itself. The main utility of Black's formula is that implied σ_B is more stable than the price itself and can give a quick explanation of the market.

In contrast to Black's formula, the field theory pricing formula is a new formula of pricing which is derived from an arbitrage-free model of the term structure of interest rates. The field theory model of interest rates has been shown many advantages in hedging treasury bonds [2]. The field theory caplet pricing formula has been introduced in Refs. [3,4]. The main advantage of modelling the forward interest rates using field theory is that there are infinitely many random variables at each instant driving the forward interest rates. Hence, we need not consider exactly correlated forward interest rates; for the field theory model the correlation of forward rates for different maturities is accurately explained by propagator of field theory in Ref. [5].

An empirical study of the field theory pricing and Black's caplet formulae is conducted in this paper. The main result in the field theory pricing formula is that the effective volatility q of a caplet is computed by a three-dimensional integration on the correlation between forward interest rates in future time. The following are three different approaches discussed in this paper for fixing the effective field theory volatility q for pricing caplets:

- The volatility function σ_H and parameters in the correlation of the field theory model, say μ , λ and η are all fitted from historical Libor data.
- The market correlator is computed directly from Libor market data.
- A parametric formula for the effective volatility q and consequently the implied volatility σ_I for the field theory model is determined from historical caplet price.² The value of σ_I is quite distinct from σ_B since σ_I is a function of future time and can be used for extrapolating to the future. In contrast, σ_B is a value that has to be computed every single day from the caplet price.

In Section 2 the caplet market and the pricing formula based on field theory and also the Black's formula are briefly reviewed. In Section 3 the empirical study is carried out, and results based on historical volatility, market correlator and implied volatility are shown and comparison with Black's formula is also given. More results on other cap instruments are discussed in Section 4. Some conclusions are drawn in Section 5. A brief review of field theory model of forward interest rate and a discussion on martingale are given in Appendices B and C, respectively.

2. A review of interest-rate caplet

The theoretical framework for the empirical study of interest rates and its derivatives is the formulation of the forward interest rates as a two-dimensional Gaussian quantum field theory; the caplet pricing formula for Libor derivatives is based on the field theory model.

This section provides a brief introduction to the cap market. The pricing formula of caplet based on field theory model is reviewed, and the industry standard Black's model is compared with the field theory formula.

Financial market's participants sometimes have to enter into financial contracts in which they pay or receive cash flows tied to some floating rate such as Libor. In order to hedge the risk caused by the Libor's variability,

²Note in contrast σ_H is obtained from the historical data of the forward interest rates.

participants often enter into derivative contracts with a fixed upper limit or lower limit of Libor at cap rate. These types of derivatives are known as interest-rate caps and floors.

A cap gives its holder a series of European call options or caplets on the Libor rate, where all caplets have the same strike price, but a different expiration dates. Typically, the expiration dates for the caplets are on the same cycle as the frequency of the underlying Libor rate.

A midcurve caplet is defined as a caplet that is exercised at time t_* , that is, before the time at which the caplet is operational. Suppose the midcurve caplet is for the Libor rate for time interval T_n to $T_n + \ell$, where ℓ is 90 days, and matures at time t_* . Let the caplet price, at time $t_0 < t_*$, be given by $Caplet(t_0, t_*, T_n)$, the domain of the underlying structure evolved in the pricing is given in Fig. 1. The payoff for the caplet is given by [6]

$$Caplet(t_0, t_*, T_n) = \ell VB(t_*, T_n + \ell)[L(t_*, T_n) - K]_+,$$

where $B(t_*, T_n + \ell)$ is the treasury bond and V is the principal for which the interest-rate caplet is defined. $L(t_*, T_n)$ is the value at time t_* of the Libor rate applicable from time T_n to $T_n + \ell$ and K is the cap rate (the strike price). Note that while the case flow on this caplet is received at time $T_n + \ell$, the Libor rate is determined at time t_* , which means that there is no uncertainty about the case flow from the caplet after Libor is set at time t_* .

From the fundamental theorem of finance the price of the $Caplet(t_0, t_*, T_n)$ is given by the expectation value of the payoff function discounting—using the spot interest rate $r(t) = f(t, t)$ —from future time t_* to present time t_0 , and yields [3]

$$Caplet(t_0, t_*, T_n) = \ell VE[e^{-\int_{t_0}^{t_*} r(t)} B(t_*, T_n + \ell)[L(t_*, T_n) - K]_+]$$

with the price of a floorlet defined by

$$Floorlet(t_0, t_*, T_n) = \ell VE[e^{-\int_{t_0}^{t_*} r(t)} B(t_*, T_n + \ell)[K - L(t_*, T_n)]_+].$$

Put-call parity relation is given by [6]

$$Caplet(t_0, t_*, T_n) - Floorlet(t_0, t_*, T_n) = \ell VB(t_0, T_n + \ell)[L(t_0, T_n) - K]. \tag{1}$$

Thus, we can get floorlet price from this put-call parity and independent derivation is not necessary. We will examine this put-call parity on the market data in next section.

An interest-rate cap of duration over longer period than the period of ℓ is made from the sum over caplets spanning the requisite time interval, namely starting from time $T_n = n\ell$ and ending at time $T_n + \ell = (n + 1)\ell$.

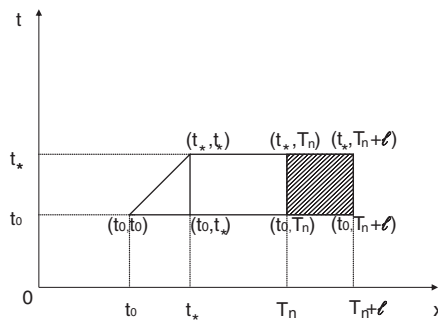


Fig. 1. Domain for underlying term structure (domain for integration of q) evolved in pricing midcurve caplet.

The price of a midcurve cap is a sum of midcurve caplets, and is given by

$$Cap(t_0, t_*) = \sum_{j=m}^n Caplet(t_0, t_*, T_j; K_j). \quad (2)$$

2.1. Field theory caplet pricing formula

A quantum field theory for forward interest rates has been introduced in Refs. [3,4], and the field theory pricing formula has been derived by Baaquie [3].

To evaluate the price of the caplet one uses a martingale measure for the evolution of the forward interest rates.³ The expectation value required for obtaining the price of caplet is evaluated using field theory; one then obtains the price of the caplet, at time $t_0 < t_*$, given by the following Black–Scholes-type formula [6]:

$$Caplet(t_0, t_*, T_n) = \tilde{V}B(t_0, T_n)[XN(d_+) - \mathcal{F}N(d_-)], \quad (3)$$

where $N(d_{\pm})$ is the cumulative distribution for the normal random variable with the following definitions⁴:

$$\begin{aligned} \mathcal{F} &= e^{-\int_{t_n}^{T_n+\ell} dx f(t_0, x)} = e^{-F_0}, \\ d_{\pm} &= \frac{1}{q} \left[\ln\left(\frac{X}{\mathcal{F}}\right) \pm \frac{q^2}{2} \right] \end{aligned} \quad (4)$$

and

$$\begin{aligned} X &= \frac{1}{1 + \ell K}, \quad \tilde{V} = (1 + \ell K)V, \\ q^2 &= q^2(t_0, t_*, T_n) \\ &= \int_{t_0}^{t_*} dt \int_{T_n}^{T_n+\ell} dx dx' \sigma(t, x) D(x, x'; t) \sigma(t, x'). \end{aligned} \quad (5)$$

The domain for integration is given in Fig. 1. Note that q is the effective volatility for the caplet pricing formula. Observe that the propagator for forward rates is required for pricing the caplet. Ultimately, the pricing formulae for caplets and floorlets stem from the volatility function $\sigma(t, x)$ and correlation parameters μ, λ, η contained in the Lagrangian for the forward interest rates, as well as the initial interest-rates term structure.

The pricing formula at the money case for a normal caplet ($T_* = T_n$) has $X = e^{\mathcal{F}}$, and is given by

$$Cap(t_0, t_*) = VB(t_0, t_*)[N(d_+) - N(d_-)], \quad (6)$$

where

$$d_{\pm} = \pm \frac{q}{2}. \quad (7)$$

We will use this formula for a comparison with the Black's model.

2.2. Black's formula for interest-rate caps

Black's model for interest-rate cap is briefly reviewed in order to compare it with the field theory model. Black's formula is based on the assumption that the spot interest rate is a log normal random variable. The payoff function for Black's formula at time t_* is [7]

$$g_B(t_*) = \frac{\ell V}{1 + \ell R(t_*)} (R(t_*) - R)_+. \quad (8)$$

³A brief discussion of the field theory model and the martingale used here is given in Appendix.

⁴Note it recovers the normal caplet when setting $T_n = t_*$.

$R(t_*)$ is the 3-month LIBOR rate at the beginning of the quarter at time t_* , and R is the strike price on the interest rate.

Black’s formula for the value of the cap at time $t_0 < t_*$ is [1]

$$Cap_B(t_0, t_*, R) = \frac{V\ell}{1 + \ell f(t_0, t_*)} B(t_0, t_*) [f(t_0, t_*) N(d_+^B) - RN(d_-^B)], \tag{9}$$

where

$$d_{\pm}^B = \frac{1}{\sigma_B \sqrt{t_* - t_0}} \left[\ln \frac{f(t_0, t_*)}{R} \pm \frac{\sigma_B^2 (t_* - t_0)}{2} \right]$$

and $f(t_0, t_*)$ denotes the ℓ period forward rate from t_* to $t_* + \ell$. At time $t_* = n\ell$, $f(t_0, t_*) = L(t_0, t_*)$.

When at the money, $f(t_0, t_*) = R$, the pricing is given by

$$Cap_B(t_0, t_*, R) = \frac{f(t_0, t_*)}{1 + \ell f(t_0, t_*)} V\ell B(t_0, t_*) [N(d_+^B) - N(d_-^B)] \tag{10}$$

with

$$d_{\pm}^B = \frac{\sigma_B \sqrt{(t_* - t_0)}}{2}.$$

Clearly the price of a cap derived from an arbitrage free model, as in Eq. (3), and Black’s formula do not agree. Even at the money, since the prefactor of two pricing formulae are different, the effective volatility q in field theory formula and σ_B are not equal. We can normalize the prefactor by multiplying Black’s formula by $(1 + \ell f(t_0, t_*)) / \ell f(t_0, t_*)$, and then the two pricing formulae are exactly the same and with q equal to σ_B . We change the cap rate K to compare the normalized pricing formula away from the money. The comparison is shown in Fig. 2, and the field theory model price is shown more clearly in Fig. 3. We can see that only at the money when cap rate $K = 0.02$, the two pricing formulae have the same result. The caplet pricing for the two formulae in general are not equal and deviate quite rapidly when K is no longer at the money.

We will see later that q has many advantages over σ_B . Most importantly, the effective volatility q is obtained in the field theory model, as computed in Eq. (5), from the underlying forward interest rates that are common to all Libor-based options.

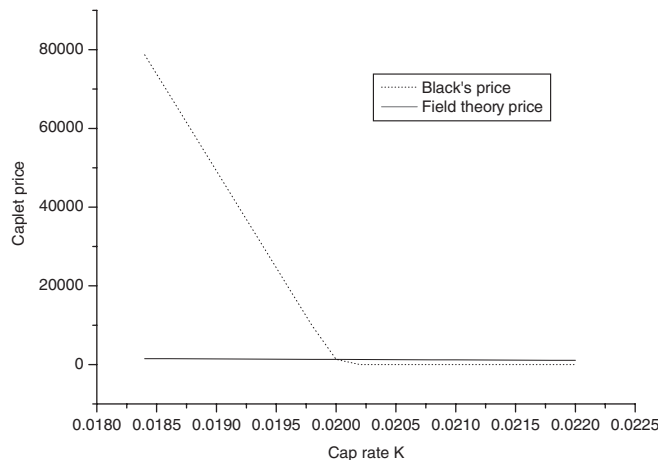


Fig. 2. Caplet price from normalized Black’s formula and field theory formula versus cap rate K . Libor is given at 0.02. The caplet is at the money when $K = 0.02$.

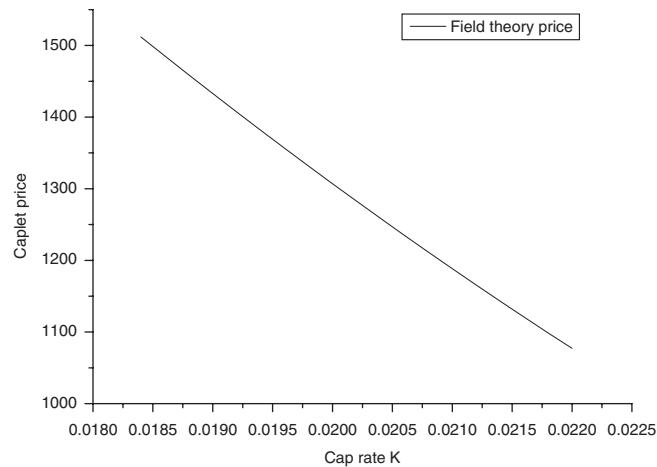


Fig. 3. Caplet price from field theory formula versus cap rate K . Libor is given at 0.02. The caplet is at the money when $K = 0.02$.

3. Empirical pricing of field theory caplet price

In this section, the cap data and underlying forward interest-rates data are discussed, put-call parity is empirically tested. An empirical study of three different approaches for implementing the field theory caplet pricing is carried out and the results are discussed.

3.1. Data

We analyze the price of the option on Eurodollar future contracts expiring on 13 December 2004 with a strike price of 98. Daily price is from 7th March 2003 to 28th May 2004. All the prices are presented with interest rate in basis points (100 basis points = 1% annual interest rate) and have to multiplied by the notional value of 1 million Dollars.

The put option of this data is equivalent to the caplet price with fixed maturity date, say 13 December 2004. The importance of put-call parity for pricing and choosing numeraire has been emphasized in Baaquie [6]; we examine market prices for the caplet and floorlet to see how the put-call parity is obeyed by the market. From Eq. (1) for put-call parity, for the case $t_* = T_n = 13$ December 2004, we form a portfolio

$$\Pi(t_0) = \text{Caplet}(t_0, T_n) - \text{Floorlet}(t_0, T_n) - \ell VB(t_0, T_n + \ell)(L(t_0, T_n) - K). \quad (11)$$

The market value of the portfolio directly from data is plotted in Fig. 4, and shows that the market automatically obey put-call parity very well. The deviation of $\Pi(t_0)$ is negligible compared to the price of a caplet. Hence, there are no arbitrage opportunities in the pricing of caplets and floorlets. We price the caplets from the pricing formula, the floorlet's price being given by the put-call parity relation.

The following are three distinct approaches to fitting the field theory caplet pricing formula.

3.2. Parameters for the field theory caplet price using historical Libor

For the field theory pricing formula for the daily prices of the caplets, we need as input the daily initial term structure, the input volatility function and parameters μ , λ and η for propagator. Daily fit of the volatility function and propagator parameters can be derived by a daily moving average on 60-day Libor rates history.⁵ For the sake of simplicity, from Refs. [3,8] we take volatility function to depend only on future time, namely

$$\sigma(t, x) = \sigma(x - t).$$

⁵The moving average can be any length of history days, and 60 days is chosen as most convenient.

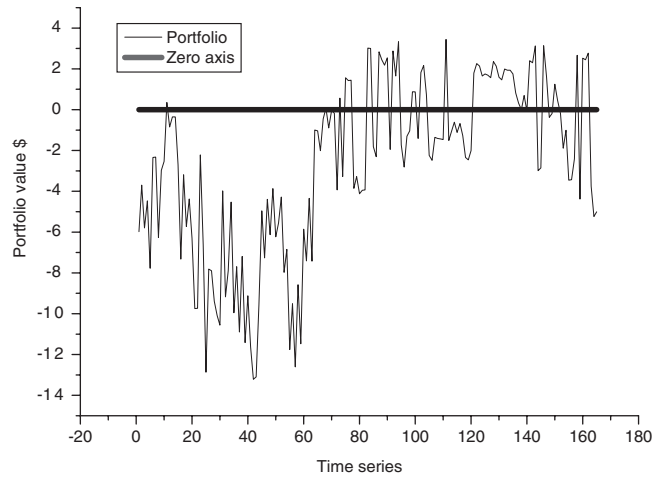


Fig. 4. Value of portfolio $\Pi(t_0)$ as in Eq. (11) with notional value 1 million versus time t_0 (12.9.2003–7.5.2004).

Although this assumption cannot be indefinitely extended, it can be valid for up to 3 years [8] which is enough for our empirical study. Thus, we only need to do the parametric fit once, then use these parameter for the whole data set projected to 1.5 years in the future. It should be noted that one can always do the parametric fit more frequently to get more accurate results.

Since the field theory is defined on a domain $x \geq t$, the propagator satisfies

$$D(t, x, x') = D(x - t, x' - t). \tag{12}$$

We fit a parametric curve for the effective volatility [9] using historical data from the prices of Libor before May 2003.

More precisely, the forward interest rates that we use to fix input volatility and propagator are data from the Eurodollar futures which covers from 4th May 1998 to 29th April 2003; the length of the data set is 1256 trading days for daily prices of Libor 7 years into the future. For $\ell = 3$ months, we have

$$L(t, T) = \frac{e^{\int_t^{T+\ell} f(t,x) dx} - 1}{\ell} \tag{13}$$

$$\simeq f(t, T). \tag{14}$$

Hence, we use the Libor as being exactly equal to the forward rates, and then select a moving average over the last 63 days, from 29th January 2003 to 29th April 2003 since these carry the most relevant information.

Following Bouchaud and Matacz [9], we give a parametric formula for volatility and fix the parameters from the data which we select and the function is given as follows⁶:

$$\sigma_H(\theta) = 0.00055 - 0.00026 \exp(-0.71826(\theta - \theta_{min})) + 0.0006(\theta - \theta_{min}) \exp(-0.71826(\theta - \theta_{min})), \tag{15}$$

where $\theta_{min} = 3$ months.

Baaquie and Bouchaud in Ref. [5] have determined the empirical values of the three parameters μ, λ, η for the stiff Lagrangian by fitting the propagator to market correlation and have demonstrated that this formulation is able to accurately account for the phenomenology of Libor interest-rate dynamics. By using the same stiff propagator and fitting techniques, the three parameters are fixed by Libor forward rates data from 29th January 2003 to 29th April 2003 as follows. The fits for volatility of Libor is given in Fig. 5. And the

⁶We use σ_H to present volatility from historical Libor rates.

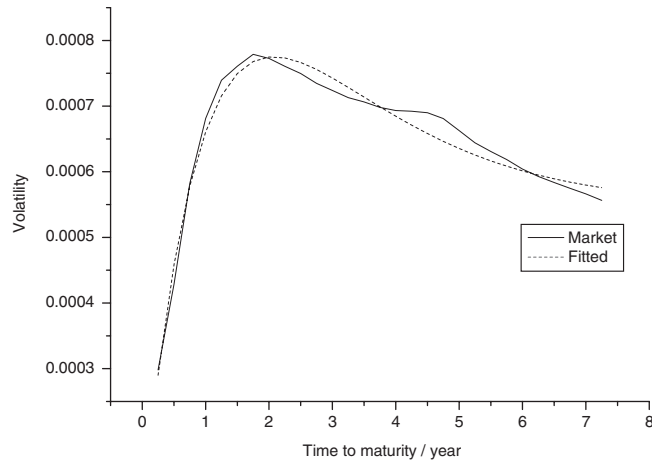


Fig. 5. Volatility of Libor forward rates σ (year^{-3/2}) versus time to maturity, both from data and from formula with fitted parameters. Data select from 29.1.2003 to 29.4.2003. Normalized root mean square error is 2.76%.

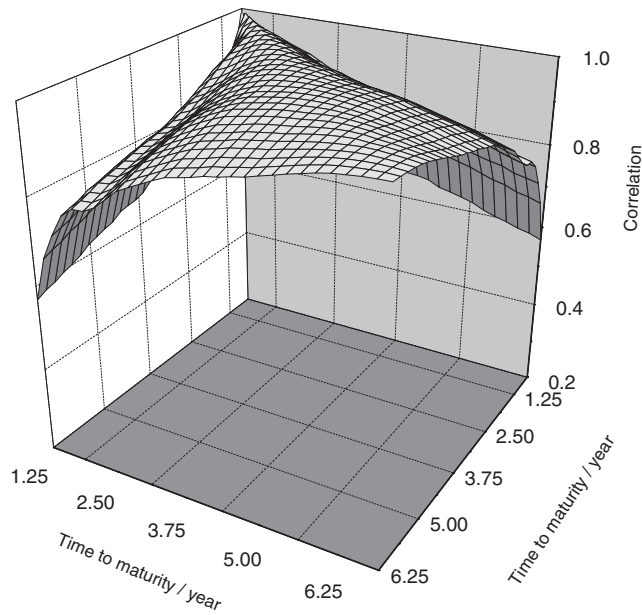


Fig. 6. Correlation of Libor forward rates versus time to maturity, data selected from 29.1.2003 to 29.4.2003.

correlation is given in Fig. 6, the graph shows that the underlying forward rates are not perfectly correlated, the parameters are given as follows

λ	μ	b	η	Root mean square error for the entire fit
16.578657/year	8.0761/year	1.376644	0.044127	1.09%

Before pricing the caplet with all the information we have, one more thing needs to be noted. Denoting by $\langle \dots \rangle$ the expectation value of a stochastic quantity, we have from Baaquie [3] for the connected correlator

$$\langle (\delta f(t, \theta))^2 \rangle_c \equiv \langle (\delta f(t, \theta))^2 \rangle - \langle \delta f(t, \theta) \rangle \langle \delta f(t, \theta) \rangle = \varepsilon \sigma_H^2(\theta) D(\theta, \theta) \tag{16}$$

setting $\varepsilon = \frac{1}{260}$, where 260 is the trading days in 1 year.

To be able to compare the volatilities of different Gaussian models, we can re-scale the field $A(t, \theta)$ so that $D(\theta, \theta) = 1/\varepsilon$. The re-scaled frame yields the usual definition of volatility of the forward rates, given as follows:

$$\langle (\delta f(t, \theta))^2 \rangle_c = \sigma_H^2(t, \theta), \tag{17}$$

where note the ε has been absorbed in the correlator. Thus, when we use the correlator as an input in real calculation, D without normalization has to be used; hence, for the effective volatility q we have

$$\begin{aligned} q^2 &= q^2(t_0, t_*, T_n) \\ &= \frac{1}{\varepsilon} \int_{t_0}^{t_*} dt \int_{T_n}^{T_n+\ell} dx dx' \sigma_H(t, x) \tilde{D}(x, x'; t) \sigma_H(t, x'). \end{aligned} \tag{18}$$

Using as input the initial forward rates curve and volatility as well as correlation and from the pricing formula from Eq. (3), we can obtain the empirical field theory caplet price. We see from Fig. 7 that the computed caplet price matches the market value very well, the normalized root mean square error is 17.39%.

3.3. Market correlator for field theory caplet price

Note the parametric fit for σ and propagator finally yields the market correlator given by

$$M(t, x, x') = \sigma(t, x) D(x, x'; t) \sigma(t, x').$$

Although these parameters give us insights on the field theory model itself we can also obtain M directly from data without fitting any of the parameters. Note we have

$$M(t, x, x') = \frac{1}{\varepsilon} \langle \delta f(t, x) \delta f(t, x') \rangle_c = M(x - t, x' - t) \tag{19}$$

and this in turn is sufficient to determine the effective volatility q .

Libor data can be interpolated since it only depends on $\theta = x - t$. Furthermore, caplets are instruments that have only a short duration, being based on the 3-month Libor. We can re-express the formula for q^2 in the following manner:

$$q^2 = \int_{t_0}^{t_*} dt \int_{T_n-t}^{T_n+\ell-t} d\theta d\theta' M(\theta, \theta'). \tag{20}$$

The integration on time requires us to have future M ; since M is a function of future time θ and θ' , we can shift the average block of $M(\theta, \theta')$ back to its historical values. For calculating q^2 we need to do one integration on t , which is reduced to a summation of the average value on different blocks of history

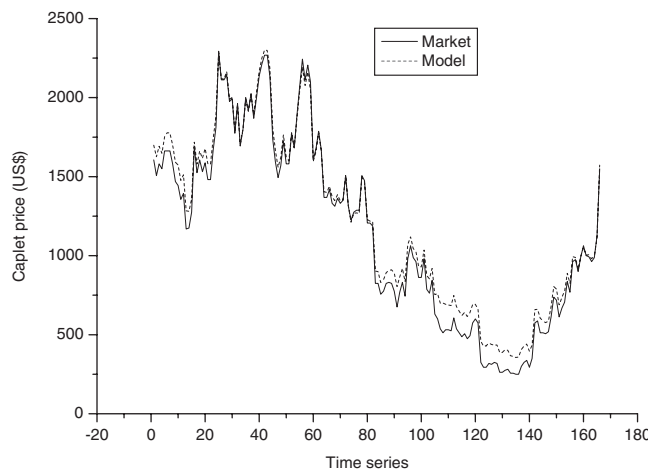


Fig. 7. Caplet price which matures on 12.12.2004 versus time t_0 (12.9.2003–7.5.2004), from market and model computed based on historical volatility and correlation which is fitted from historical Libor rates, normalized root mean square error = 17.39%.

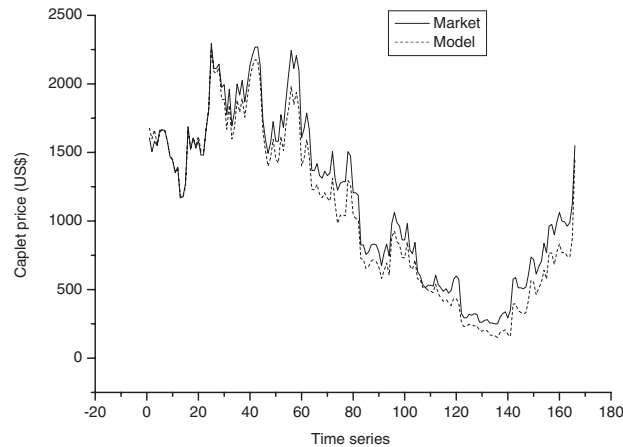


Fig. 8. Caplet price which matures on 12.12.2004 versus time t_0 (12.9.2003–7.5.2004), both market and model price with effective volatility q computed directly from Libor rates. Normalized root mean square error = 17.89%.

Libor data; the difference among the parallelogram blocks is only a horizontal shift and all of them end at time t_0 .

Furthermore, since the Libor data we have are expressed in Eq. (13) as integration of forward interest rates, we can save two integrations and directly use the Libor data without approximating by the forward interest rates as in Eq. (14). We can hence price the caplet by directly obtaining q^2 from the data; this is more efficient and more accurate. Market data yields q^2 by the following correlator:

$$q^2 = \int_{t_0}^{t^*} dt \langle \delta Y(t, T_n) \delta Y(t, T_n) \rangle_c, \quad (21)$$

where

$$\begin{aligned} Y(t, T_n) &= \int_{T_n}^{T_n+\ell} dx f(t, x) \\ &= \ln(1 + \ell L(t, T_n)). \end{aligned} \quad (22)$$

The computational cap price is given by Fig. 8 where we can see it again matches the market price very well, the normalized root mean square error is 17.89%.

3.4. Market fit for effective volatility from caplet price

Recall that we have computed q both by fitting the parameters of the field theory model and directly by using the market correlator—both of which use historical Libor data. Another alternative is that of directly fitting q from the market caplet prices, thus yielding the implied volatility σ_I . In contrast σ_H is obtained from $\langle (\delta f(t, \theta))^2 \rangle_c$.

The first approximate fit (both accurate and simple) for the effective volatility is to fit q as a linear function $q = b\theta$ and then implied volatility is a square root function of future time.⁷ The linear fit in future time $\theta = x - t$ obviously cannot explain the phenomena of implied volatility since it blows up as time goes into future. However, for the market price of caplet over only a short duration, and square root volatility provides a very good fit. For time far into the future we directly fit the implied volatility with an exponential formula, as in Eq. (15). The fitting is for the first 100 days in the same data set, say from 12.9.2003 to 4.2.2004. The best fit

⁷Fitting effective volatility is much easier than fitting correlation from option price data. Furthermore, the impact of changing correlation is insignificant compared with changing effective volatility since a caplet only involves the correlation between two neighboring forward interest rates within the range of a single caplet, and hence over a maximum future time difference of 90 days; to a good approximation within a single $\ell = 90$ days, $D(x, x') \simeq 1$.

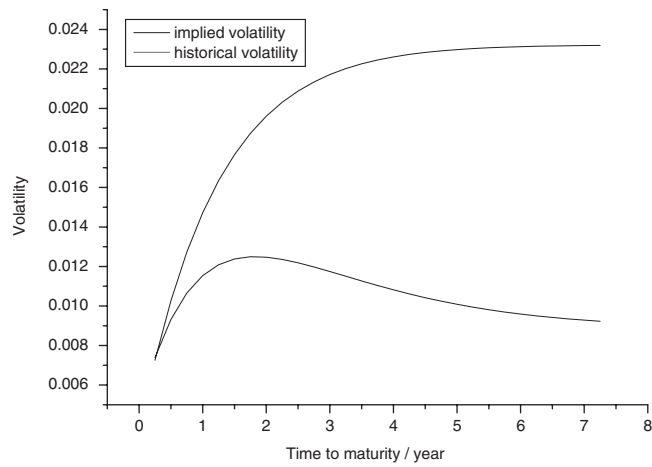


Fig. 9. Implied volatility σ_I (year^{-3/2}) fitted from caplet data (12.9.2003–4.2.2004) versus time to maturity.

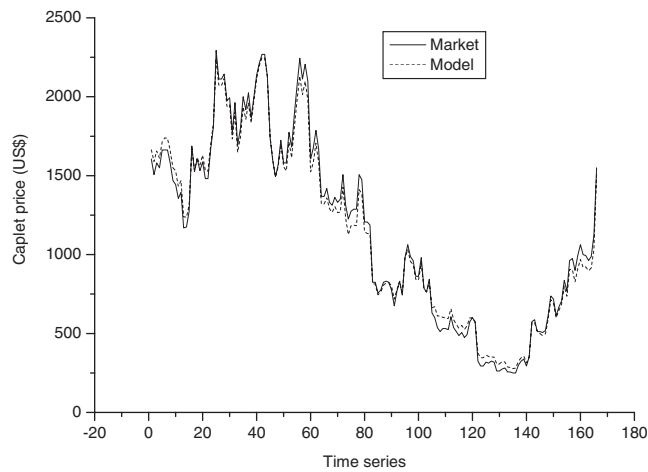


Fig. 10. Caplet price which matures on 12.12.2004 versus time t_0 (12.9.2003–7.5.2004), both market and model with implied volatility fitted directly from first 100-days caplet price. Normalized root mean square error = 6.67%.

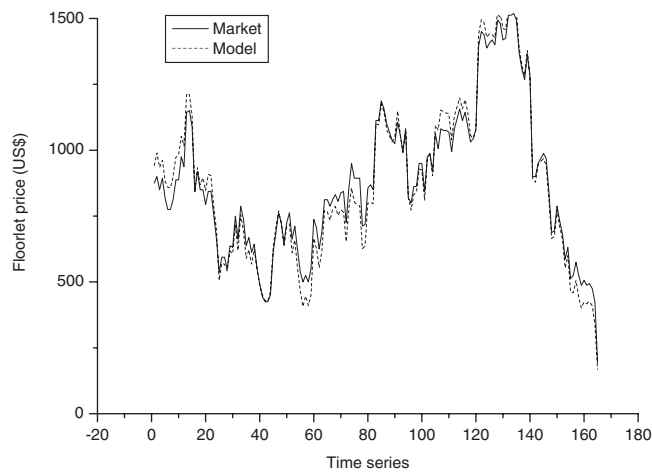


Fig. 11. Floorlet price which matures on 12.12.2004 versus time t_0 (12.9.2003–7.5.2004), both market and model with implied volatility fitted directly from first 100-days floorlet price. Normalized root mean square error = 7.9%.

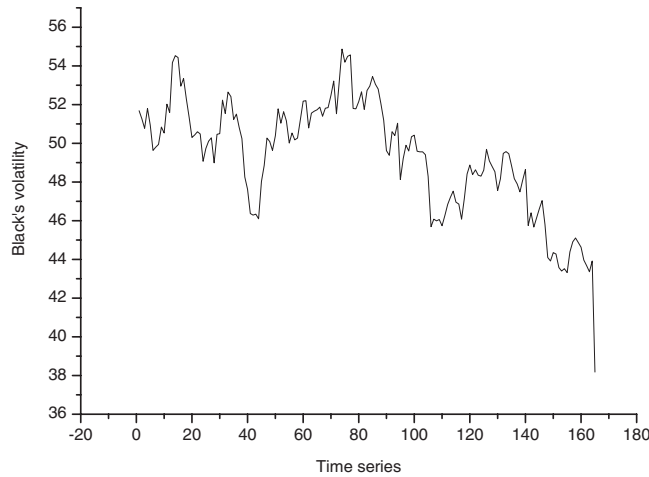


Fig. 12. Market Black volatility σ_B for caplet which matures on 12.12.2004 versus time t_0 (12.9.2003–7.5.2004).

for σ_I is given in Fig. 9, and is the following:

$$\sigma_I(\theta) = 0.00144 - 0.00122 \exp(-0.71826(\theta - \theta_{min})) + 0.00014(\theta - \theta_{min}) \exp(-0.71826(\theta - \theta_{min})). \quad (23)$$

We use the effective volatility fitted using the first 100 days to price the whole 168-days cap using the field theory caplet pricing formula (3). Results are shown in Fig. 10 with normalized root mean square error 6.67% and also floorlet price is shown in Fig. 11 with normalized root mean square error 7.9%.

One can always do a daily moving fit for q to improve the accuracy of the calculation, and the technique used is the same as here. We only fit q once and use the fitted volatility to price the whole time series that we select. The empirical result shows that the fit is already good enough to show some important features of the field theory caplet pricing formula.

Given below are the root mean square errors of the above three approaches for fitting the field theory caplet price:

	σ_H from Libor	Market correlator	σ_I from Caplet
Normalized root mean Square error in caplet price (%)	17.39	17.89	6.67

3.5. Comparison of field theory caplet price with Black's formula

Recall that the price of a caplet is equivalent to an effective value for Black's implied volatility σ_B , and one obtains an implied volatility everyday from the price; σ_B is shown in Fig. 12. The shape of Black's implied volatility is very irregular and cannot be fitted well by any formula. No prediction can be made for the future value of Black's volatility and hence we cannot extrapolate it to the future and make a prediction for the price of a caplet for future time.

By comparison with the Black's formula, we see that the field theory model yields a non-trivial result. The effective volatility q and thus the implied volatility σ_I are fitted from caplet prices by the field theory formula as shown in Figs. 9 and 13, and can be used for pricing caplet prices in the future since σ_I is only a function of $x - t$.

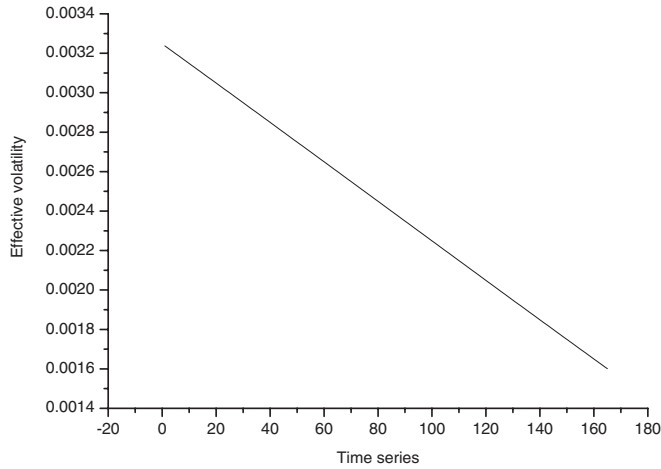


Fig. 13. Effective volatility q for caplet which matures on 12.12.2004 versus time t_0 (12.9.2003–7.5.2004).

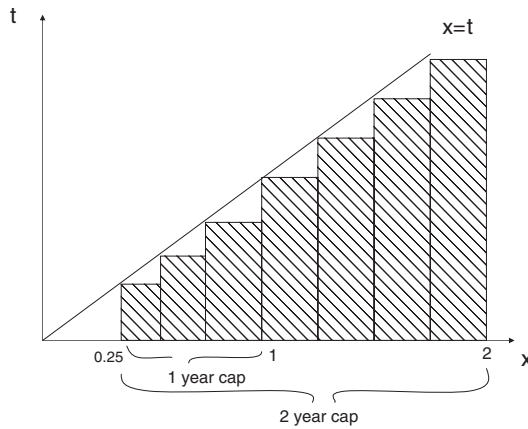


Fig. 14. Domain for 1 and 2 year cap. For 1 year cap, three caplets involved. For 2 year cap, seven caplets evolved.

4. Pricing an interest-rate cap

We apply the field theory caplet pricing formula to the pricing of interest-rate cap and study fixed maturity cap data. We analyze 494 trading days market cap price which mature 1, 2 and 3 years in the future. We try to price the same cap using the field theory formula and compare it to the market price.

Fixed maturity cap is a sum of caplets shown as below:

$$Cap(t_0, T_N) = \sum_{n=1}^{N-1} Caplet(t_0, T_n). \tag{24}$$

The caplet price is based on 3-month Libor, and the first caplet matures at 3 months. A 1 year cap can be expressed as a sum of three caplets, a 2 year cap is a sum of seven caplets; the domain of the cap can be seen in Fig. 14.

We first use the field theory implied volatility σ_I , fitted by the fixed maturity date cap on the same period, to price the 2 and 3 year cap. The computed price shown in Figs. 15 and 16. Since now there is a new instrument everyday, one can always improve accuracy by fitting moving lattice effective volatility⁸ directly from the 1,2,3 year cap.

⁸Only Libor time q need to be fitted.

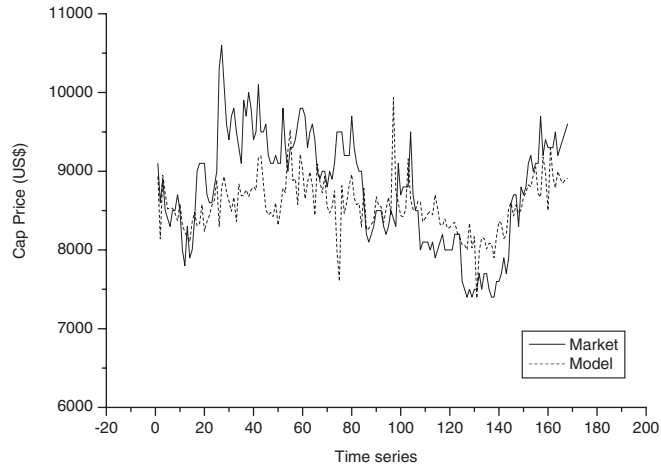


Fig. 15. Cap price which matures 2 years in the future versus time t_0 (12.9.2003–7.5.2004), both market and model with effective volatility fitted by caplet price. Normalized root mean square error = 6.7%.

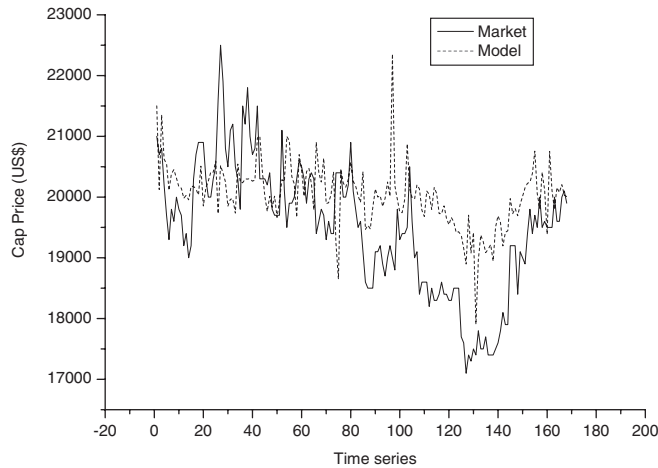


Fig. 16. Cap price which matures 3 years in the future versus time t_0 (12.9.2003–7.5.2004), both market and model with effective volatility fitted by caplet price. Normalized root mean square error = 5.54%.

We can also price the cap by the historical Libor data using Eq. (21). Results are shown in Fig. 17. The normalized root mean square errors in cap price are⁹

	$Cap(t_0,2)$ from σ_I	$Cap(t_0,3)$ from σ_I	$Cap(t_0,3)$ from market correlator
Normalized root mean Square error in cap price (%)	6.7	5.54	5.59

5. Conclusion

Libor-based caps and floors are important financial instruments for managing interest rate risk. However, the multiple payoffs underlying these contracts complicates their pricing as the Libor term structure dynamics

⁹The values of normalized root mean squares here are smaller than those for caplets in Figs. 7, 8 and 10 since RMS is normalized by the price and the cap prices here are much bigger than those for caplets. But the original points of the graph here are not zero and the differences between fit and observation are stretched and seems inconsistent with the normalized root mean square.

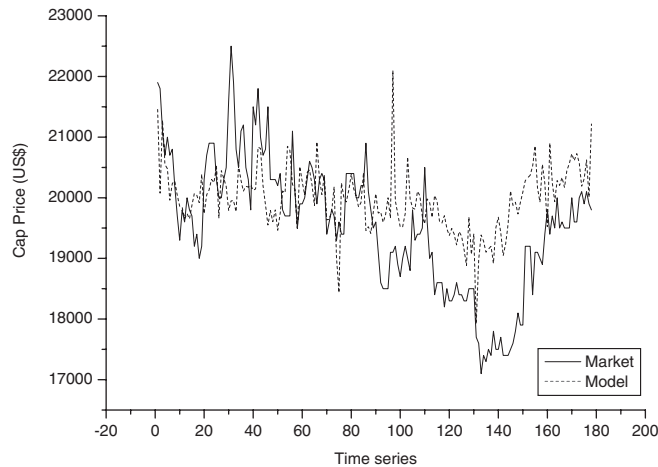


Fig. 17. Cap price which matures 3 years in the future versus time t_0 (12.9.2003–21.5.2004), both market and model with effective volatility computed directly from historical Libor rates. Normalized root mean square error = 5.59%.

are not perfectly correlated. A field theory model which allows for imperfect correlation between every Libor maturity overcomes this difficulty while maintaining model parsimony.

We did an empirical study of the field theory pricing formula of interest-rates caplets, and used three alternative approaches—with all of the three approaches showing satisfactory results. Unlike Black’s model, the effective volatility q for the field theory caplet pricing formula can be derived from the underlying historical Libor rate, and hence the field theory caplet pricing formula yields a *prediction* for the caplet price: given the input Libor data, the field theory model generates the daily caplet price. More importantly, it can be used to price other Libor-based options; in contrast Black’s formula is just a (non-linear) representation of the market price, with a one-to-one relation between market price and implied Black volatility σ_B .

Historical caplet prices also served to obtain a best fit for the effective volatility q needed for the field theory caplet pricing; the best q obtained from historical data can be used to generate long future caplet prices.

We also empirically studied different interest-rate cap data, and demonstrated the accuracy of the field theory model for pricing interest rate caps.

Acknowledgments

We thank Jiten Bhanap for many useful discussions. The data in our empirical tests were generously provided by Bloomberg, Singapore.

Appendix A. Review of Libor and caplet

We will briefly discuss the Libor and caplet in this Appendix.

A.1. Libor

We briefly review the main features of the Libor market for the readers who are unfamiliar with this financial instrument. The discussion follows [3].

Eurodollar refers to US\$ bank deposits in commercial banks outside the US. These commercial banks are either non-US banks or US banks outside the US. The deposits are made for a fixed time, the most common being 90- or 180-day time deposits, and are exempt from certain US government regulations that apply to time deposits inside the US.

The Eurodollar deposit market constitutes one of the largest financial markets. The Eurodollar market is dominated by London, and the interest rates offered for these US\$ time deposits are often based on Libor, the *London Interbank Offer Rate*. The Libor is a simple interest rate derived from a Eurodollar time deposit of 90 days. The minimum deposit for Libor is a par value of \$1 000 000. Libor are interest rates for which commercial banks are willing to lend funds in the interbank market.

Eurodollar futures contracts are amongst the most important instrument for short-term contracts and come to dominate this market. The Eurodollar futures contract, like other futures contracts, is an undertaking by participating parties to loan or borrow a fixed amount of principal at an interest rate fixed by Libor and executed at a specified future date.

Eurodollar futures as expressed by Libor extends to up to 10 years into the future, and hence there are underlying forward interest rates driving all Libor with different maturities. For a futures contract entered into at time t for a 90-day deposit of \$1 million from future time T to $T + \ell$ ($\ell = 90/360$ years), the principal plus simple interest that will accrue—on the maturity—to an investor long on the contract is given by

$$P + I = 1 + \ell L(t, T),$$

where $L(t, T)$ is the (annualized) 3-month (90-day) Libor. Let the forward interest rates for the 3-month Libor be denoted by $f(t, x)$. One can express the principal plus interest based on the compounded forward interest rates and obtain

$$P + I = e^{\int_T^{T+\ell} dx f(t,x)},$$

hence the relationship between Libor and its forward rates is given by

$$L(t, T) = \frac{e^{\int_T^{T+\ell} dx f(t,x)} - 1}{\ell}. \quad (25)$$

Sometimes one may need to assume that the Eurodollar futures Libor prices are equal to the forward rates. More precisely, from Eq. (25)

$$L(t, T) \simeq f(t, T) + O(\ell). \quad (26)$$

A.2. Caplet

We illustrate Black's formula for pricing cap by working out a real life example.

Consider a contract that caps the interest rate on a \$1 million loan for 3 months with Libor rate. The contract is written on $t = 13.9.2003$ which mature on $t_* = 12.12.2004$ with a cap rate R given by 2%. The Libor $L(t, t_*)$ at 13.9.2003 for 3-month Eurodollar deposit from 12.12.2004 to 12.3.2005 is given by 2.95% per annum. The bond price $B(t, t_*)$ is 0.984. Referring to Section 2.2 Eqs. (9) and (10), we have

$$f(t, t_*) = L(t, t_*) = 0.0295,$$

then

$$d_+^B = \frac{1}{0.5168 * \sqrt{1.25}} \left[\ln \frac{0.0295}{0.02} + \frac{0.5168^2 * 1.25}{2} \right] = 0.527,$$

$$d_-^B = d_+^B - 0.5168 * \sqrt{1.25} = -0.0508.$$

Thus,

$$\begin{aligned} \text{Caplet}(t, t_*, 0.02) &= \$ \frac{1\,000\,000 * 0.25}{1 + 0.25 * 0.0295} * 0.984 [0.0295 * N(0.527) - 0.02 * N(-0.0508)] \\ &= \$1587.655. \end{aligned}$$

Appendix B. A brief summary of field theory model of forward interest rate

The field theory of forward rates is a general framework for modelling the interest rates that allows for a wide choice of evolution equation for the interest rates.

The forward interest rates $f(t, x)$ are the interest rates, fixed at time t , for an instantaneous loan at future time $x > t$. The price at time t_* of a treasury bond that matures at some future time $T > t_*$ —denoted by $B(t, T)$ —is defined in terms of the forward interest rates as follows:

$$B(t, T) = e^{-\int_t^T dx f(t,x)}. \tag{27}$$

Suppose a treasury bond $B(t_*, T)$ is going to be issued at some time $t_* > t$, and will also expire at time T ; the forward price of the treasury bond is the price that one pays at time t to lock-in the delivery of the bond when it is issued at time t_* , and is given by

$$F(t, t_*, T) = e^{-\int_{t_*}^T dx f(t,x)}. \tag{28}$$

Let $A(t, x)$ be a two-dimensional quantum field, driving the time evolution of forward interest rates $f(t, x)$, defined by

$$\frac{\partial f(t, x)}{\partial t} = \alpha(t, x) + \sigma(t, x)A(t, x), \tag{29}$$

where $\alpha(t, x)$ is the drift of the forward interest rates that will be fixed by a choice of numeraire and $\sigma(t, x)$ is the volatility that is fixed from the market [3]. One is free to choose the dynamics of how the quantum field $A(t, x)$ evolves.

Integrating Eq. (29) yields

$$f(t, x) = f(t_0, x) + \int_{t_0}^t dt' \alpha(t', x) + \int_{t_0}^t dt' \sigma(t', x)A(t', x), \tag{30}$$

where $f(t_0, x)$ is the initial forward interest-rates term structure that is specified by the market.

Following Baaquie and Bouchaud [5], the Lagrangian that describes the evolution of instantaneous forward rates is defined by three parameters μ, λ, η , and is given by¹⁰

$$\mathcal{L}[A] = -\frac{1}{2} \left\{ A^2(t, z) + \frac{1}{\mu^2} \left(\frac{\partial A(t, z)}{\partial z} \right)^2 + \frac{1}{\lambda^4} \left(\frac{\partial^2 A(t, z)}{\partial z^2} \right)^2 \right\}, \tag{31}$$

where market (psychological) future time is defined by $z = (x - t)^\eta$.

The Lagrangian in Eq. (31) contains a squared Laplacian term that describes the stiffness of the forward rate curve. Baaquie and Bouchaud [5] have determined the empirical values of the three constants μ, λ, η , and have demonstrated that this formulation is able to accurately account for the phenomenology of interest-rate dynamics. Ultimately, all the pricing formulae for caplets and floors stem from the volatility function $\sigma(t, x)$ and correlation parameters μ, λ, η contained in the Lagrangian, as well as the initial term structure.

The action $S[A]$ of the Lagrangian is defined as

$$S[A] = \int_{t_0}^\infty dt \int_0^\infty dz \mathcal{L}[A]. \tag{32}$$

All financial instruments of the interest rates are obtained by performing a path integral over the (fluctuating) two-dimensional quantum field $A(t, z)$. The expectation value for an instrument, say $F[A]$, is denoted by $\langle F[A] \rangle \equiv E[F[A]]$ and is defined by the functional average over all values of $A(t, z)$, weighted

¹⁰More complicated non-linear Lagrangians have been discussed in Refs. [3,4].

by the probability measure e^S/Z . Hence,

$$\langle F[A] \rangle \equiv E[F[A]] \equiv \frac{1}{Z} \int DAF[A] e^{S[A]}, \quad Z = \int DA e^{S[A]}. \quad (33)$$

The quantum theory of the forward interest rates is defined by the generating (partition) function [3] given by

$$\begin{aligned} Z[h] &= E[e^{\int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t,z)A(t,z)}] \equiv \langle e^{\int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t,z)A(t,z)} \rangle \\ &\equiv \frac{1}{Z} \int DA e^{S[A] + \int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t,z)A(t,z)} \\ &= \exp\left(\frac{1}{2} \int_{t_0}^{\infty} dt \int_0^{\infty} dz dz' h(t,z)D(z,z';t)h(t,z')\right) \end{aligned} \quad (34)$$

which follows from the correlator of the $A(t, x)$ quantum field given by

$$\langle A(t, x)A(t', x') \rangle = E[A(t, x)A(t', x')] = \delta(t - t')D(x, x'; t). \quad (35)$$

Appendix C. A brief discussion of martingale

The price of any financial instrument in the future has to be discounted by a numeraire to obtain its current price. The freedom of choosing a numeraire results from the fact that for every numeraire there is a compensating drift such that the price of any traded instrument is independent of the numeraire. ‘Numeraire invariance’ is an important tool in creating models for the pricing of financial instruments.

In the original Heath, Jarrow, and Morton model [10], the martingale measure is defined by discounting treasury bonds denoted as $B(t, T)$ by the money market account $R(t, t_*)$, defined as

$$R(t, t_*) = e^{\int_t^{t_*} r(t) dt}, \quad (36)$$

for the spot rate of interest denoted by $r(t)$. In contrast, in this paper all computations are carried out using the Libor measure for which Libor rates evolve as martingales. In other words, for $t_* > t$,

$$L(t, T_n) = E_L[L(t_*, T_n)]. \quad (37)$$

Following the material in Baaquie [6], the drift $\alpha_L(t, x)$ that corresponds to the Libor martingale condition is given by

$$\alpha_L(t, x) = -\sigma(t, x) \int_{T_n}^x dx' D(x, x'; t) \sigma(t, x'), \quad T_n \leq x < T_{n+\ell}. \quad (38)$$

As proved in Baaquie [6], a money market numeraire entails more complex calculations but arrives at identical prices as the one obtained using the Libor measure. In this paper, the subscript L is suppressed with all expectations performed with respect to the Libor measure.

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