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Physica A 381 (2007) 285–316
PHYSICA A
www.elsevier.com/locate/physa

Pricing American options for interest rate caps and coupon bonds in quantum finance

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Received 14 November 2006; received in revised form 29 January 2007
Available online 5 March 2007

Abstract

American option for interest rate caps and coupon bonds are analyzed in the formalism of quantum finance. Calendar time and future time are discretized to yield a lattice field theory of interest rates that provides an efficient numerical algorithm for evaluating the price of American options. The algorithm is shown to hold over a wide range of strike prices and coupon rates. All the theoretical constraints that American options have to obey are shown to hold for the numerical prices of American interest rate caps and coupon bond options. Non-trivial correlation between the different interest rates are efficiently incorporated in the numerical algorithm. New inequalities are conjectured, based on the results of the numerical study, for American options on interest rate instruments.

Keywords: American option; Quantum finance; Cap and coupon bond option

1. Introduction

American options for interest rate caps and for fixed-income securities are amongst the most widely traded financial instruments. An accurate and arbitrage free pricing of American interest rate options has far reaching applications. American options for interest rates are rather complex instruments since at any moment in time, there are a large number of future interest rates that exist in the market. All of the interest rates evolve randomly and have strong correlations with the other interest rates. In principle, interest rate instruments, are described *at every instant* by infinitely many degrees of freedom (random variables).

In the simple case of a European option on equity, the Black–Scholes equation can be explicitly solved to obtain an analytical formula for the price of the option [1]. When one considers other financial derivatives that allow anticipated early exercise or depend on the history of the underlying assets, numerical approaches need to be used. Appropriate numerical procedures have been developed in the literature to price exotic financial derivatives on equity with path-dependent features, as discussed in detail in Ref. [1]. These procedures involve the use of Monte Carlo simulations, binomial tree (and their improvements) and finite difference methods.

The pricing of European and American options for the interest rates is far more complicated than for equity options. In order to price interest derivatives, one needs to model the underlying interest rates dynamics.

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2007.02.054

The leading model at present for modelling interest rates and its derivatives is the HJM (Heath, Jarrow and Morton)-model and BGM model; for the N-factor model, the interest rates at every instant are driven by N random variables [1,2]. Numerical techniques for pricing American interest rates options [2,12] are all based on the generalization of the binomial tree approach.

Baaquie [3] has developed the formalism of quantum finance to model non-trivial correlations between forward interest rates with different maturities as a parsimonious alternative to the existing interest rate theories in finance, in particular to the HJM-model [2,4]. Interest rate derivatives are shown in this paper to have a natural numerical algorithm for their pricing that directly follows from the quantum finance formulation of the forward rates.

In quantum finance, the stochastic forward interest rates are averaged over all their possible values to evaluate interest rate options and other derivatives; the averaging over the stochastic field is mathematically identical to the averaging in quantum field theory. In effect, from a mathematical point of view, the forward interest rate is a two-dimensional (stochastic) quantum field. Hence in quantum finance one uses the techniques of quantum field theory for modelling the interest rates. An efficient algorithm is developed in this paper for obtaining the price of the American option using the formalism of quantum finance.

To price American options for equity an efficient computational algorithm, using path integrals, has been developed by Montagna et al. [5] and is reviewed in Appendix A. The quantum field theory describing the forward interest rates is discretized and yields a lattice field theory model; an algorithm that generalizes the path integral approach of Montagna et al. [5] to the case of interest rate options is obtained using the lattice field theory.

In Section 2 the field theory model of forward interest rates is reviewed. In Section 3 the field theory pricing formula for European caplets and coupon bond options are reviewed and American options for interest rate instruments are defined. Lattice field theory of interest rate is defined in Section 4. The tree structure of forward interest rates is discussed in Section 5. Details of the numerical algorithm and the programming codes are given in Section 6. Numerical results for caplet and coupon bond option are discussed in Sections 7 and 8. A new inequality for American option of caplet and coupon bond is conjectured in Section 9. In Section 10 some conclusions are drawn. A brief review of the path integral algorithm for pricing American options on equity is given in Appendix.

2. Field theory model of forward interest rates

The field theory of forward rates is a general framework for modelling the interest rates that allows for a wide choice of evolution equation for the interest rates, and in particular allows one to model strongly correlated forward interest rates.

The forward interest rates $f(t,x)$ are the interest rates, fixed at time t, for an instantaneous loan at future times $x>t$. The price, at time t, of a Treasury Bond that pays a pre-fixed sum, of say \$1, when it matures at some future time $T>t$, denoted by $B(t,T)$, is defined in terms of the forward interest rates as follows:

$$B(t,T) = \exp\left\{-\int_t^T \mathrm{d}x f(t,x)\right\}. \tag{1}$$

$B(t,T)$ is also called a zero coupon bond. The forward bond price for bond to be issued at future time t_* and maturing at time T namely $B(t_*,T)$, is given by

$$F(t_0,t_*,T) = \exp\left\{-\int_{t_*}^T \mathrm{d}x f(t_0,x)\right\}.$$

A coupon bond on a principal L, which matures at time T, pays fixed dividends (coupons) a_i at times $T_i, i=1,2,\ldots,N$. The value of the coupon bond at time $t_*<T_i$ is given by [1]

$$\sum_{i=1}^N a_i B(t_*,T_i) + LB(t_*,T) = \sum_{i=1}^N c_i B(t_*,T_i), \tag{2}$$

where for simplicity of notation the final payment is included in the sum by setting $c_i = a_i; c_N = a_N + L$, and with the time of maturity of the coupon bond given by $T = T_N$.

Coupon and zero coupon bonds, forward bonds and so on are also called fixed-income securities.

Let $f(t,x)$ be a two-dimensional quantum field; following Baaquie and Bouchaud [6], the Lagrangian that describes the evolution of instantaneous forward interest rates is defined by three parameters μ, λ, η, and for $\partial f(t,x)/\partial t \equiv \dot{f}(t,x)$, is given by

$$\mathscr{L}[f] = -\frac{1}{2}\left[\left\{\frac{\dot{f}(t,x)-\alpha(t,x)}{\sigma(t,x)}\right\}^2 + \frac{1}{\mu^2}\left\{\frac{\partial}{\partial x}\left(\frac{\dot{f}(t,x)-\alpha(t,x)}{\sigma(t,x)}\right)\right\}^2 + \frac{1}{\lambda^4}\left\{\frac{\partial^2}{\partial x^2}\left(\frac{\dot{f}(t,x)-\alpha(t,x)}{\sigma(t,x)}\right)\right\}^2\right], \tag{3}$$

where $\alpha(t,x)$ is the drift of the forward interest rates that is determined by the choice of numeraire, and $\sigma(t,x)$ is the volatility that is fixed from the market [3].

Baaquie and Bouchaud [6] have determined the empirical values of the three constants μ, λ, ν, and have demonstrated that this formulation is able to accurately account for the phenomenology of interest rate dynamics. Ultimately, all the pricing formulae for interest rate instruments stems from the volatility function $\sigma(t,x)$ and correlation parameters μ, λ, ν contained in the Lagrangian, as well as the initial term structure $f(t_0, x)$.

The action $S[f]$ of the Lagrangian is defined as

$$S[f] = \int_{t_0}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}x \mathscr{L}[f]. \tag{4}$$

Doing an integration by parts in the maturity direction using the Neumann boundary conditions [3] yields, from Eqs. (3) and (4), the action

$$S = -\frac{1}{2}\int_{t_0}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}x \left(\frac{\dot{f}(t,x)-\alpha(t,x)}{\sigma(t,x)}\right) D^{-1}(x,x',t) \left(\frac{\dot{f}(t,x)-\alpha(t,x)}{\sigma(t,x)}\right). \tag{5}$$

All expectation values, denoted by $E[..]$, are evaluated by integrating over all possible values of the quantum field $f(t,x)$. The quantum theory of the forward interest rates is defined by the generating (partition) function [3] for the following combination of the field $[\dot{f}(t,x) - \alpha(t,x)]/\sigma(t,x)$ since it is this combination that will appear in the American option. Hence the generating function is given by

$$\begin{aligned} Z[h] &= E\left[\exp\left\{\int_{t_0}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}x h(t,x)\left[\frac{\dot{f}(t,x)-\alpha(t,x)}{\sigma(t,x)}\right]\right\}\right] \\ &\equiv \frac{1}{Z}\int Df \exp\left\{S[f] + \int_{t_0}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}x h(t,x)\left[\frac{\dot{f}(t,x)-\alpha(t,x)}{\sigma(t,x)}\right]\right\} \\ &= \exp\left(\frac{1}{2}\int_{t_0}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}x\, \mathrm{d}x' h(t,x) D(x,x';t) h(t,x')\right). \end{aligned} \tag{6}$$

The prices of interest rates instruments are obtained by performing a path integral over all possible values of the (fluctuating) two-dimensional quantum field $f(t,x)$, weighted by the probability measure e^S/Z. The expectation value for an instrument, say $F[f]$, is defined by the following:

$$E(F[f]) \equiv \frac{1}{Z}\int Df F[f] \mathrm{e}^{S[f]}, \quad Z = \int Df \mathrm{e}^{S[f]}. \tag{7}$$

3. Pricing interest rate derivatives

Caps, floors, swaptions and coupon bond options are interest rate derivatives that are widely used in the financial markets. In this section caps, caplets, floors, floorlets and the coupon bond option are defined. Caps are instruments that provide a maximum interest rate payments that the holder of the cap needs to pay for some specified period of time. Interest rate floors ensure a minimum rate of interest payments for the holders of the floor. Coupon bond is a fixed-income security that provides for regular payments of 'coupons' at prefixed intervals. Options on coupon bonds are widely traded and their pricing is a fairly complicated and nontrivial problem. Swaptions are a special case of coupon bond options [4], and hence need not be independently studied.

Midcurve options, analyzed in this section, are options that mature before the instrument becomes operational. For example a caplet may cap interest rates for a duration of three months say one year in the future, and a midcurve option on such a caplet can have a maturity time only six months, hence expiring six months before the instrument becomes operational. Similarly a midcurve option on a coupon bond may mature in say six months time with the bond starting to pay coupons only a year from now.

Midcurve options are widely traded in the market and hence need to be studied. In the numerical studies of both the caplets and coupon bond options the midcurve option will not be priced; instead, for simplicity, only options that mature when the instrument becomes operational are studied. It should be noted that all the numerical procedures used in this paper can be generalized in a straight forward manner to the midcurve case.

Every numeraire chosen for discounting the future cash flows of a financial instrument yields a martingale evolution for the traded instrument. The pricing of derivatives hinges on the martingale property of traded financial instruments [4]. The European caplet and coupon bond options at time t_0 are computed using the forward measure numeraire $B(t,\tau)$ [7]. The value of τ is fixed and is chosen to suit the instrument. For any traded financial instrument $\mathcal{I}$, the martingale property yields

$$\frac{\mathcal{I}(t_0,\tau)}{B(t_0,\tau)} = E_F\left[\frac{\mathcal{I}(t_*,\tau)}{B(t_*,\tau)}\right], \tag{8}$$

where $\mathcal{I}(t_*,\tau)$ is the payoff function at maturity time t_*.

For a midcurve caplet that matures at t_* and becomes operational at time T, the payoff function is given by $\mathcal{I}(t_*,\tau) = Caplet(t_*,t_*,T)$; the numeraire is chosen to be $B(t,\tau) = B(t,T)$. For a midcurve coupon bond option, maturing at t_*, the payoff function is given by $\mathcal{I}(t_*,\tau) = (\sum_{j=1}^{N} c_j F(t_i,t_*,T_j) - K)_+$ and the numeraire is chosen to be $B(t,\tau) = B(t,t_*)$.

3.1. Price of European interest rate caps

Market interest rates, in general, constantly fluctuate and are called floating interest rates. A typical measure of the floating interest rate is the daily Libor [3], denoted by $L(t,T)$, which is the interest rate agreed upon at time t for a loan at future time $T>t$. An interest rate *caplet* is an option that puts a maximum ceiling, say K, to the interest rate for a period of $\ell = 90$ days from T to $T+\ell$ and is exercised if Libor goes above the stipulated ceiling. The price of a midcurve caplet, issued at time t_0 and maturing at time $t_* \in [t_0,T]$, is denoted by $Caplet(t_0,t_*,T)$.[1]

Let the principal amount be equal to ℓV, and the caplet rate be K. The *payoff function* of the caplet is given by [7]

$$Caplet(t_*,t_*,T) = \ell V B(t_*,T+\ell)[L(t_*,T)-K]_+ \tag{9}$$

$$= \tilde{V} B(t_*,T)(X-F_*)_+, \tag{10}$$

where

$$L(t_*,T) = \frac{e^{\int_T^{T+\ell} dx f(t_*,x)} - 1}{\ell}, \quad F_* = F(t_*,T,T+\ell) = \exp\left\{-\int_T^{T+\ell} dx f(t_*,x)\right\},$$

$$X = \frac{1}{1+\ell K}, \quad \tilde{V} = (1+\ell K)V.$$

[1] A European midcurve caplet can be exercised only at maturity time t_*.

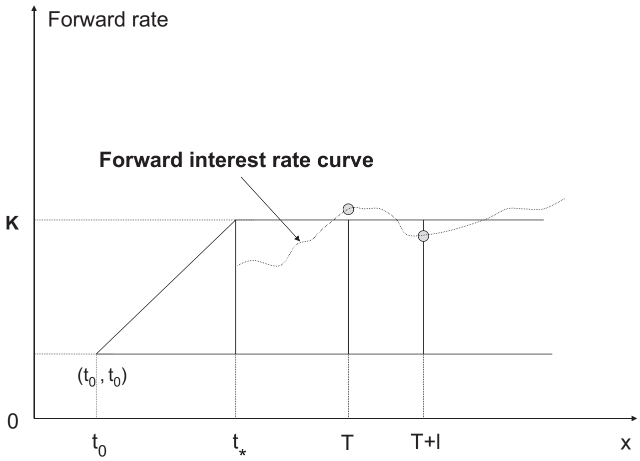

Fig. 1. Diagram representing a caplet $\ell VB(t_*, T+\ell)[L(t_*, T) - K]_+$. During the time interval $T \leqslant t \leqslant T+\ell$, the borrower holding a *caplet* needs to pay only K interest rate, regardless of the values of forward interest rate curve during this period.

Fig. 1 shows how a caplet provides a cutoff to the maximum interest rate that a borrower holding a caplet will need to pay.

The payoff function for a floorlet is given by

$$Floorlet(t_*, t_*, T) = \tilde{V}B(t_*, T)(F_* - X)_+$$

and ensures the lender holding the floorlet option receives a minimum rate of K for the interest payments.

The European caplet at time t_0 is computed using the forward measure with numeraire $B(t, T)$ and Eq. (8) yields

$$\frac{Caplet(t_0, t_*, T)}{B(t_0, T)} = E_F\left[\frac{Caplet(t_*, t_*, T)}{B(t_*, T)}\right] \tag{11}$$

$$\Rightarrow Caplet(t_0, t_*, T) = \tilde{V}B(t_0, T)E_F(X - F_*)_+. \tag{12}$$

Fig. 2 shows the domain over which the midcurve caplet is defined.

One obtains a closed form of the European caplet price by evaluating the expectation value using field theory. At time $t_0 < t_*$ the caplet price is given by the following Black–Scholes-type formula [7]

$$Caplet(t_0, t_*, T) = \tilde{V}B(t_0, T)[XN(d_+) - FN(d_-)], \tag{13}$$

where $N(d_\pm)$ is the cumulative distribution for the normal random variable with the following definitions[2]:

$$F = \exp\left\{-\int_T^{T+\ell} \mathrm{d}x f(t_0, x)\right\},$$
$$d_\pm = \frac{1}{q}\left[\ln\left(\frac{X}{F}\right) \pm \frac{q^2}{2}\right], \tag{14}$$

and

$$q^2 = q^2(t_0, t_*, T) = \int_{t_0}^{t_*} \mathrm{d}t \int_T^{T+\ell} \mathrm{d}x\, \mathrm{d}x' \sigma(t, x) D(x, x'; t) \sigma(t, x'). \tag{15}$$

The caplet formula, derived by Baaquie [3], has been empirically studied in Ref. [8] and shown to be fairly accurately.

[2]Note one recovers the normal caplet result by setting $t_* = T$.

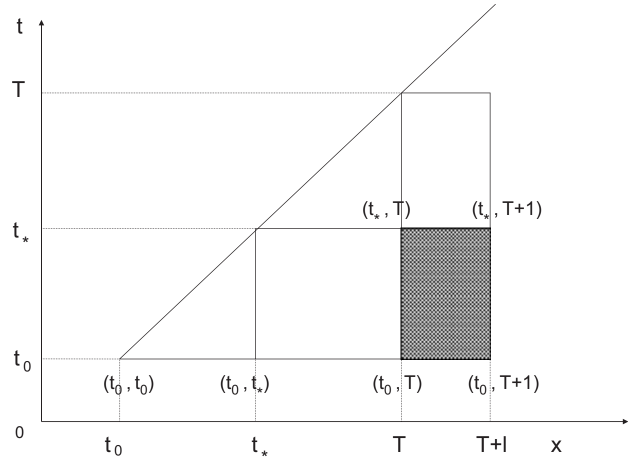

Fig. 2. The domain of the midcurve caplet in the xt plane; the payoff $\tilde{V}B(t_0, T)E_F(X - F_*)_+$ is defined at time t_*. The shaded portion shows the domain of the forward interest rates that define the price $Caplet(t_0, t_*, T)$ for a midcurve caplet.

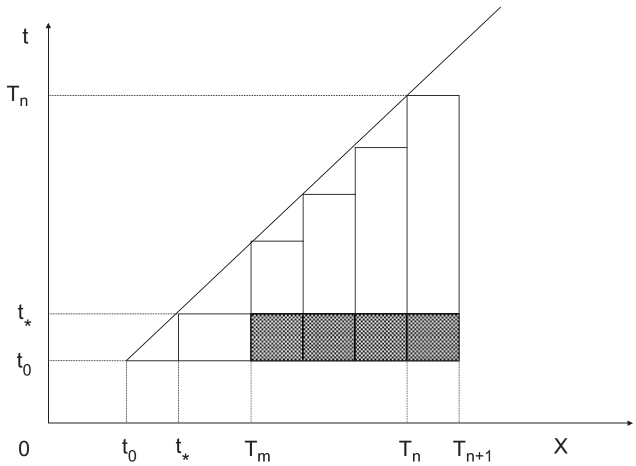

Fig. 3. The domain of the midcurve interest rate cap $Cap(t_0, t_*) = \sum_{j=m}^{n} Caplet(t_0, t_*, T_j; K_j)$, defined from future time T_m to time T_n in terms of the portfolio of midcurve caplets. The shaded portion indicates the domain of the forward interest rated required for the pricing of the midcurve $Cap(t_0, t_*)$.

The numerical result for the European caplet price obtained in Section 7 will be compared to the analytical formula given in Eq. (13) to ascertain the accuracy of the numerical result.

An interest rate cap with a duration over a longer period is made from the sum over caplets spanning the requisite time interval. Consider a midcurve cap, to be exercised at time t_*, with cap starting from time $T_m = m\ell$ and ending at time $T_{n+1} = (n+1)\ell$; its price is given by

$$Cap(t_0, t_*) = \sum_{j=m}^{n} Caplet(t_0, t_*, T_j; K_j). \tag{16}$$

Fig. 3 shows the structure of the an interest cap in terms of it's constituent caplets.

It follows from above that the price of an interest cap only requires the prices of interest rate caplets. Hence, in effect, one needs to obtain the price of a single caplet for pricing interest rate caps.

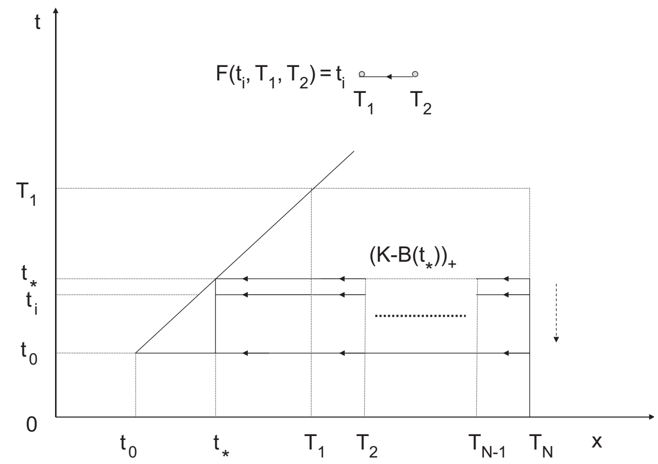

Fig. 4. The payoff of the coupon bond option $S(t_*) = (\sum_{i=1}^{N} c_i B(t_*, T_i) - K)_+$ is represented by the horizontal line at t_*. The successive horizontal lines show the (non-discounted) value of the payoff for $t < t_*$.

3.2. Approximate price of European coupon bond option

An approximate expression for the European coupon bond options has been derived in Ref. [9], and is one of the key formulae in checking the accuracy of the numerical algorithms developed in this paper; the main results of the derivation are discussed below.

The payoff function $S(t_*)$ for a coupon bond call option maturing at time t_* and with strike price K, is given by

$$S(t_*) = \left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+ . \tag{17}$$

Fig. 4 shows the domain on which a coupon bond is defined.

The forward measure with numeraire $B(t, t_*)$ is chosen for pricing the coupon bond option. From Eq. (8) the martingale property yields at time $t_0 < t_*$, the following coupon bond option price:

$$\frac{C(t_0, t_*, K)}{B(t_0, t_*)} = E_F \left[\frac{S(t_*)}{B(t_*, t_*)} \right] \tag{18}$$

$$\Rightarrow C(t_0, t_*, K) = B(t_0, t_*) E_F \left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+ . \tag{19}$$

The option price has been derived in Ref. [9] and yields the results given below[3]

$$C(t_0, t_*) = B(t_0, t_*) \sqrt{\frac{A}{2\pi}} \left[\frac{B}{6A^{3/2}} X + \frac{C}{24A^2} (X^2 - 1) + \frac{1}{72} \frac{B^2}{A^3} (X^4 - 6X^2 + 3) \right] e^{-(1/2)X^2}$$
$$+ B(t_0, t_*) \sqrt{\frac{A}{2\pi}} I(X) + O(\sigma^4), \tag{20}$$

[3]The error function is given by

$$\Phi(u) = \frac{2}{\sqrt{\pi}} \int_0^u \mathrm{d}W \, e^{-W^2}.$$

where

$$I(X) = \mathrm{e}^{-(1/2)X^2} - \sqrt{\frac{\pi}{2}}X\left[1 - \Phi\left(\frac{X}{\sqrt{2}}\right)\right], \quad X = \frac{K - F}{\sqrt{A}},$$

$$F \equiv \sum_{i=1}^{N} J_i, \quad J_i \equiv c_i F_i, \quad F_i \equiv F_i(t_0, t_*, T_i) = \exp\left\{-\int_{t_*}^{T_i} \mathrm{d}x f(t_0, x)\right\}.$$

Note F_i are the forward bond prices of $B(t_*, T_i)$.

The coefficients in the option price are given in Ref. [9] by

$$A = \sum_{ij=1}^{N} J_i J_j \left[G_{ij} + \frac{1}{2}G_{ij}^2\right] + O(G_{ij}^3),$$

$$B = 3\sum_{ijk=1}^{N} J_i J_j J_k G_{ij} G_{jk} + O(G_{ij}^3),$$

$$C = 16\sum_{ijkl=1}^{N} J_i J_j J_k J_l G_{ij} G_{jk} G_{kl} + O(G_{ij}^4). \tag{21}$$

The market correlator G_{ij} of the forward bond prices is given by

$$\begin{aligned} G_{ij} &\equiv G_{ij}(t_0, t_*, T_i, T_j; \sigma) \\ &= \int_{t_0}^{t_*} \mathrm{d}t \int_{t_*}^{T_i} \mathrm{d}x \int_{t_*}^{T_j} \mathrm{d}x' \sigma(t, x) D(t, x, x') \sigma(t, x') \\ &= G_{ji}\text{: real and symmetric.} \end{aligned} \tag{22}$$

This pricing formula has been empirically studied in Ref. [10] and has been demonstrated to be fairly accurately. The results of this formula are compared in Section 8 with those from the algorithm to test the accuracy of the numerical prices.

3.3. American caplet and coupon bond options

The American option has the same payoff function as the European option, but with the additional feature that it can be exercised at any time before the expiration day. To define the numerical algorithm the time interval $[t_0, t_*]$ is discretized; let $t_0 = 0$. It is assumed that an option can be exercised at any time before maturity but only at the discrete times. Since one has to evolve the payoff function backwards in time, for the numerical algorithm it is more convenient to label time *backwards*, with the origin of the time lattice being placed at t_*, the maturity of the payoff function. Hence define lattice time by $t_i = t_* - (i-1)\varepsilon = (M - i + 1)\varepsilon, i = 1, 2, \ldots, M+1$, where $t_1 = t_*$; present time $t_0 = 0$ yields for lattice time $t_{M+1} = 0$ and hence fixes $t_* = M\varepsilon$. In other words the option can only be exercised at time $t_1 = M\varepsilon, t_2 = M\varepsilon - \varepsilon, t_3 = M\varepsilon - 2\varepsilon, \ldots, t_M = M\varepsilon - (M-1)\varepsilon = \varepsilon$.

Let $C(t_i, t_*)$ denote the price of both caplet and coupon bond option, the third argument T in $Caplet(t_i, t_*, T)$ being suppressed. In the forward measure the ratio $C(t_i, t_*)/B(t_i, t_*)$ is a martingale. From Eqs. (11) and (18), the initial trial value of the American option at later time t_{i+1}, denoted by $g_I(t_{i+1})$, is given from the option price at time t_i by the martingale property as follows:

$$\begin{aligned} g_I(t_{i+1}) &= E_F\left[\frac{C(t_i, t_*)}{B(t_i, t_*)}\right], \\ &\Rightarrow g_I(t_{i+1}) = E_F[g(t_i)]. \end{aligned} \tag{23}$$

The subscript I in $g_I(t_{i+1})$ denotes the initial trial value of the American option at t_{i+1}. The trial option price is compared with the payoff function (divided by the appropriate numeraire) and yields the actual value of the option at time t_{i+1} being equal the maximum of the two [1].

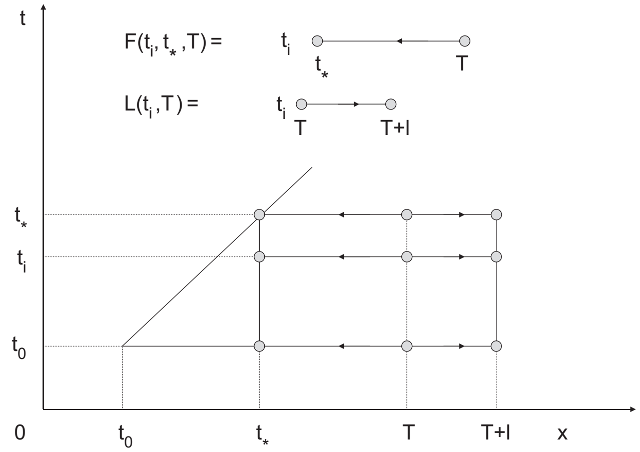

Fig. 5. The (non-discounted) payoff function $VF(t_i, t_*, T)(X - F(t_i, T, T + \ell))_+ / B(t_i, t_*)$ for the caplet at intermediate time $t_i \in [t_0, t_*]$.

3.3.1. Caplet

The scaled caplet payoff is given by

$$V \frac{F(t_i, t_*, T)(X - F(t_i, T, T + \ell))_+}{B(t_i, t_*)}, \quad F(t_i, T, T + \ell) = \exp\left\{ - \int_T^{T+\ell} \mathrm{d}x f(t_i, x) \right\}. \tag{24}$$

The important point to note that the form of the payoff does not change with time; the discounting factor at t_* that appears in the payoff at maturity, namely the bond $B(t_*, T)$ is changed into the forward bond $F(t_i, t_*, T)$ as one moves to an intermediate time t_i; there is no additional discounting factor. The American option price at time t_{i+1} is equal to the maximum of the initial trial option value $g_I(t_{i+1})$ and the payoff function at time t_{i+1}; hence

$$\frac{C(t_{i+1}, t_*)}{B(t_{i+1}, t_*)} = g(t_{i+1}) = \mathrm{Max}\left[g_I(t_{i+1}), V \frac{F(t_{i+1}, t_*, T)(X - F(t_{i+1}, T, T + \ell))_+}{B(t_{i+1}, t_*)} \right]. \tag{25}$$

Fig. 5 shows the forward interest rates that define the caplet payoff function at different times t_*, t_i, t_0.

3.3.2. Coupon bond

The scaled coupon bond payoff is given by

$$\frac{(\sum_{j=1}^{N} c_j F(t_i, t_*, T_j) - K)_+}{B(t_i, t_*)}, \quad F(t_i, t_*, T_j) = \exp\left\{ - \int_{t_*}^{T_j} \mathrm{d}x f(t_i, x) \right\}, \tag{26}$$

where, as is the case for the interest rate caplet, in the payoff function at intermediate time $t_i \in [t_0, t_*]$ the bond price at time t_*, namely $B(t_*, T_j)$ has been replaced, at time t_i, by the forward bond price $F(t_i, t_*, T_j)$. The American option price at time t_{i+1} is given by

$$\frac{C(t_{i+1}, t_*)}{B(t_{i+1}, t_*)} = g(t_{i+1}) = \mathrm{Max}\left[g_I(t_{i+1}), \frac{(\sum_{j=1}^{N} c_j F(t_{i+1}, t_*, T_j) - K)_+}{B(t_{i+1}, t_*)} \right]. \tag{27}$$

Note the important fact that for both the caplet and coupon bond, the payoff function at each time t_i is identical to the form of the payoff function at maturity time t_*; in particular, the payoff function is not discounted when it is compared to the trial value of the option $g_I(t_i)$.

4. Lattice theory of forward interest rates

The field theory of forward interest rates is defined on the semi-infinite continuous xt plane. To obtain a numerical algorithm the xt plane needs to be discretized into a lattice consisting of a finite number of points.

The time direction, as mentioned earlier, is discretized into a lattice with spacing ε and future time direction x is also discretized into a lattice with spacing a.

Recall from Eqs. (3) and (4) that the action for continuous time and future time is given

$$S = -\frac{1}{2}\int \mathrm{d}t \int \mathrm{d}x \left[\left(\frac{\dot{f} - \alpha}{\sigma} \right)^2 + \frac{1}{\mu^2} \left(\frac{\partial}{\partial x} \left(\frac{\dot{f} - \alpha}{\sigma} \right) \right)^2 + \frac{1}{\lambda^4} \left(\frac{\partial^2}{\partial x^2} \left(\frac{\dot{f} - \alpha}{\sigma} \right) \right)^2 \right]. \tag{28}$$

The time lattice is defined as discussed in Section 3.3, and for simplicity of notation $t_0 = 0$. Future time, similar to calendar time, is labelled running *backwards*, with the origin of future time being placed at the payoff function. In other words, the continuous time and future time labels (t, x) are discretized so that lattice time is defined by $t_* - t$ and lattice future time is defined by $T - x$, where T is the maturity time of the option. The definitions below make the mapping from continuous to lattice indices more precise.

Given the trapezoidal shape of the forward rates domain defined by $x \geqslant t$, the range of discretized x_j depends on discretized time t_i. At maturity t_*, future time is taken to have N lattice points corresponding to N forward rates. Hence, for $t \in [0, t_*]$, the discretized calendar and future time are given by

$$t \to t_i = (M - i + 1)\varepsilon, \quad i = 1, 2, \ldots, M + 1, \ \ t_* = M\varepsilon,$$

$$t_i : x \to x_j = M\varepsilon + (N - j + 1)a, \quad j = 1, 2, \ldots, N + i, \ \ T - t_* = Na. \tag{29}$$

The total number of lattice sites is equal to $N(M + 1) + M(M + 1)/2$. Note for most numerical calculations one usually takes $\varepsilon = a$.

Fig. 6 shows the lattice on which the forward interest rates are defined.

To define the lattice theory, one needs to rescale the field $f(t, x)$ and all the parameters so that all the quantities that appear in the theory are dimensionless. For this reason, define the following dimensionless lattice quantities:

$$f_{mn} = af(t, x) = af((M - i + 1)\varepsilon, M\varepsilon + (N - j + 1)a),$$

$$\tilde{\alpha}_{mn} = a\varepsilon\alpha(t, x), \quad s_{mn} = \sqrt{\varepsilon a}\sigma(t, x),$$

$$\tilde{\mu} = a\mu, \quad \tilde{\lambda} = a\lambda.$$

The dimensionless field variables f_{mn} yield the following discretizations:

$$\dot{f} = \frac{1}{a\varepsilon}(f_{m,n} - f_{m+1,n}) \equiv \frac{1}{a\varepsilon}\delta_t f_{mn}, \tag{30}$$

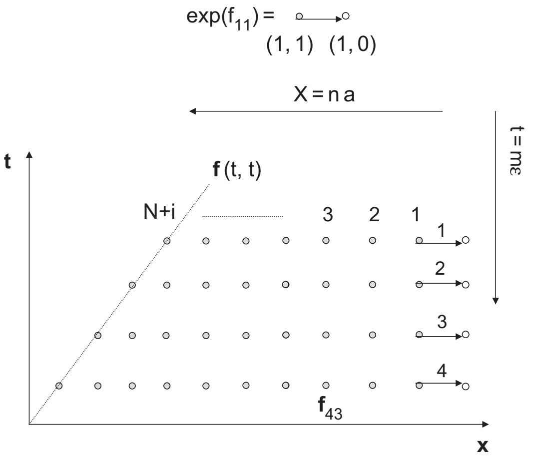

Fig. 6. Forward interest rates on the lattice, with each dot representing a forward rate $f(t, x) \to f_{ij}$; an arrow indicates that a forward rate $e^{f_{ij}}$ connects lattice (ij) to $(i, j + 1)$. The lattice points take values in the range $i = 1, 2, \ldots, M + 1$ and $j = 1, 2, \ldots, N + i$.

$$\partial_x f = \frac{1}{a^2}(f_{m,n} - f_{m,n+1}) \equiv \frac{1}{a^2}\delta_x f_{mn}. \tag{31}$$

Thus, from Eq. (28), one obtains the lattice action S, expressed completely in terms of dimensionless field variables and parameters and is given by

$$S_L = -\frac{1}{2}\sum_{m,n}\left[\left(\frac{\delta_t f - \tilde{\alpha}}{s_{mn}}\right)^2 + \frac{1}{\tilde{\mu}^2}\left(\delta_x\left(\frac{\delta_t f - \tilde{\alpha}}{s_{mn}}\right)\right)^2 + \frac{1}{\tilde{\lambda}^4}\left(\delta_x^2\left(\frac{\delta_t f - \tilde{\alpha}}{s_{mn}}\right)\right)^2\right]. \tag{32}$$

Doing an integration by parts, the action in Eq. (32) yields

$$S_L = -\frac{1}{2}\sum_{m=1}^{M+1}\sum_{n=1}^{N+m}\left(\frac{\delta_t f - \tilde{\alpha}}{s}\right)_{mn}\tilde{D}^{-1}_{m,nn'}\left(\frac{\delta_t f - \tilde{\alpha}}{s}\right)_{mn'}, \tag{33}$$

where $\tilde{D}^{-1}_{m,nn'}$ is the dimensionless inverse of the propagator with dimensionless parameters $\tilde{\mu}, \tilde{\lambda}$. The dimensionless lattice Lagrangian is given from S by the following [3]:

$$S_L = \sum_{m=1}^{M}\mathscr{L}[\mathbf{f}_{m+1};\mathbf{f}_m],$$

$$\mathscr{L}[\mathbf{f}_{m+1};\mathbf{f}_m] = -\frac{1}{2}\sum_{n,n'=1}^{N+m}\left(\frac{\delta_t f - \tilde{\alpha}}{s}\right)_{mn}\tilde{D}^{-1}_{m,nn'}\left(\frac{\delta_t f - \tilde{\alpha}}{s}\right)_{mn'}, \tag{34}$$

where $\mathbf{f}_m = (f_{m1}, f_{m2}, \ldots, f_{m,N+m})$. Note the length of the forward interest rate vector $\mathbf{f}_m$ depends on the time lattice m and has $N+m$-components; the reason being that forward interest rates are defined for all $x \geqslant t$, which on the lattice implies that the number of forward rates for a given t_i depends on t_i.

The functional integral is discretized and yields the lattice field theory of forward interest rates given by

$$Z = \int Df\, e^S \to Z_L = \tilde{\mathscr{N}}\prod_{m=1}^{M+1}\prod_{n=1}^{N+m}\int_{-\infty}^{+\infty} df_{mn} e^{S_L}, \tag{35}$$

with normalization $\tilde{\mathscr{N}}$.

The American option for the caplet is formulated on the lattice. For starters discretize the payoff function of caplet in Eq. (10); at maturity time t_* the discretized caplet is denoted by $C_1 \equiv C(t_*, t_*, T)$.

On the lattice, Libor is given by

$$1 + \ell L(t,T) = \exp\left\{\int_T^{T+\ell} dx f(t,x)\right\} \simeq \exp\left\{\frac{\ell}{a}f_{m1}\right\} = \exp\{f_{m1}\}, \tag{36}$$

where, for simplicity and because Libor data is given only on a future time lattice with spacing ℓ, one takes $\ell = a$. Hence

$$Caplet(t_*, t_*, T) = \tilde{V}B(t_*, T)(X - F_*)_+ \equiv C_1$$

$$\Rightarrow C_1 = \tilde{V}\exp\left\{-\sum_{j=1}^{N+1} f_{1j}\right\}(X - e^{-f_{11}})_+$$

$$= C_1(f_{11}, f_{12}, \ldots, f_{1,N+1}) = C_1(\mathbf{f}_1). \tag{37}$$

The payoff function is evolved backwards in time to obtain the price of the option from the payoff function. To illustrate the general procedure, consider the first step backwards; one starts from the payoff function at time $t_* = t_1$ and find the value of the option at time t_2, since recall increasing the index of lattice time ones goes backwards in time. In taking one step backwards in time, the number of independent forward interest rates on the lattice increases by one rate, starting from forward rates $\mathbf{f}_i = (f_{i,1}, f_{i,2}, \ldots, f_{i,N+i})$ and ending up with $\mathbf{f}_{i+1} = (f_{i+1,1}, f_{i+1,2}, \ldots, f_{i+1,i}, f_{i+1,N+i+1})$. Hence the option price given by $C(\mathbf{f}_i) = C_i(f_{i,1}, f_{i,2}, \ldots, f_{i,N+i})$ evolves to $C(\mathbf{f}_{i+1}) = C_{i+1}(f_{i+1,1}, f_{i+1,2}, \ldots, f_{i+1,N+i+1})$, where sometimes the notation $C_i \equiv C_i(f_{i,1}, f_{i,2}, \ldots, f_{i,N+i})$ is used.

The expression $\mathcal{N}\exp\{\mathcal{L}[\mathbf{f}_{i+1},\mathbf{f}_i]\}$ is the *pricing kernel* for interest rate options, analogous to the pricing kernel for the (simpler) case of equity given in Eq. (64). Similar to Eq. (67) for equity, the pricing kernel propagates the option price $C_i(\mathbf{f}_i)$ backwards in time and yields the option price $C_{i+1}(\mathbf{f}_{i+1})$ at earlier time t_{i+1}.

From Eq. (11), the convention being used for future lattice time is that for all t_i, $T \to x_1$, that is, the minimum value of the future lattice index x_n; hence the zero coupon bond in the payoff function is given by $B(t_i, T) \to \exp\{-\sum_{j=1}^{N+i} f_{ij}\}$ and similarly for $B(t_{i+1}, 1)$.

The pricing kernel yields, similar to Eq. (67) for the case of equity options, the option price C_{i+1} at earlier time t_{i+1} from option price C_i by taking one step backward in time, and generates the initial trial value for the option C_{i+1}. Hence, Eq. (11) yields the following result [3][4]

$$\frac{C_{i+1}}{B(t_{i+1},1)} \equiv \frac{C_{i+1}(\mathbf{f}_{i+1})}{B(t_{i+1},1)} = \mathcal{N}\int d\mathbf{f}_i\, e^{\mathcal{L}[\mathbf{f}_{i+1},\mathbf{f}_i]}\frac{C_i(\mathbf{f}_i)}{B(t_i,1)} \tag{38}$$

$$= \mathcal{N}\prod_{p=1}^{N+i}\int df_{ip}\exp\left(-\frac{1}{2}\sum_{j,k=1}^{N+i}\left(\frac{f_{i,j}-\bar{f}_{i+1,j}}{s_{ij}}\right)D^{-1}_{i;jk}\left(\frac{f_{i,k}-\bar{f}_{i+1,k}}{s_{ik}}\right)\right)\frac{C_i}{B(t_i,1)}, \tag{39}$$

where

$$\bar{f}_{i+1,j} \doteq f_{i+1,j} + \tilde{\alpha}_{i+1,j}. \tag{40}$$

Similar to the case of American option for an equity discussed in Appendix A, the interest rate dimensionless volatility s_{mn} is quite small, that is $s_{mn} \simeq 0$. Hence in the f_{ij} integrations given in Eq. (39), the path integral will be dominated by values $f_{i,j}$ that are close to $\bar{f}_{i+1,j} = f_{i+1,j} + \tilde{\alpha}_{i+1,j}$. The most accurate way for evaluating the functional integral in Eq. (39) is to Taylor expand the function $g_i = C_i/B_{i,1}$ about $\bar{f}_{i+1,j}$. A Taylor's expansion of the kernel function g_i around $\bar{f}_{i+1,1}, \ldots, \bar{f}_{i+1,i}$ yields, for $\bar{g}_i \equiv g_i(\bar{f}_{i+1,j})$, the following:

$$g_i = \bar{g}_i + \sum_{j=1}^{N+i}\frac{\partial\bar{g}_i}{\partial f_{ij}}(f_{ij}-\bar{f}_{i+1,j}) + \frac{1}{2}\sum_{j,k=1}^{N+i}\frac{\partial^2\bar{g}_i}{\partial f_j\partial f_k}(f_{ij}-\bar{f}_{i+1,j})(f_{ik}-\bar{f}_{i+1,k}) + \cdots. \tag{41}$$

The Gaussian integrations over all the forward rates f_{ij} is carried out using the following results:

$$\mathcal{N}\prod_{p=1}^{N+i}\int df_{ip}e^{\mathcal{L}[\mathbf{f}_{i+1},\mathbf{f}_i]} = 1, \quad \mathcal{N}\prod_{p=1}^{N+i}\int df_{ip}e^{\mathcal{L}[\mathbf{f}_{i+1},\mathbf{f}_i]}(f_{ij}-\bar{f}_{i+1,j}) = 0,$$

$$\mathcal{N}\prod_{p=1}^{N+i}\int df_{ip}e^{\mathcal{L}[\mathbf{f}_{i+1},\mathbf{f}_i]}(f_{ij}-\bar{f}_{i+1,j})(f_{ik}-\bar{f}_{i+1,k}) = s_{ij}\tilde{D}_{i;jk}s_{ik}. \tag{42}$$

Hence, Eqs. (39) and (42) yield the result that

$$C_{i+1} = B_{i+1,1}\left[\bar{g}_i + \frac{1}{2}\sum_{j,k=1}^{N+i}\frac{\partial^2\bar{g}_i}{\partial f_{i,j}\partial f_{ik}}s_{i,j}\tilde{D}_{i;jk}s_{ik}\right] + \cdots. \tag{43}$$

Recall the function $g_i = g_i[\mathbf{f}_i]$ depends on the vector $\mathbf{f}_i = (f_{i,1}, f_{i,2}, \ldots, f_{i,N+i})$. The value of $\bar{g}_i = \bar{g}_i[\bar{f}_{i+1,j}]$, that is, on the entire forward rate tree at time t_{i+1}. Since $g_i[\mathbf{f}_i]$ is being differentiated with respect to only two of the components, namely $f_{i,j}, f_{i,k}$, only these (two) components will be explicitly indicated, with the rest of the components in $g_i[\mathbf{f}_i]$ being suppressed.

The second derivative of g_i is numerically estimated using the symmetric second order difference. The spacing with δ is taken to be $O(s)$, as dictated by the Lagrangian in Eq. (39). The symmetric second order

[4]In terms of the Hamiltonian H of the forward interest rates the option price C_{i+1} is given by

$$\frac{C_{i+1}}{B_{i+1,1}} = \prod_{j=1}^{N+i}\int df_{ij}\langle f_{i+1,1}, f_{i+1,2}, \ldots, f_{i+1,N+i}|e^{-\varepsilon H}|f_{i,1}, f_{i,2}, \ldots, f_{i,N+i}\rangle\frac{C_i}{B_{i,1}}.$$

See Ref. [3] for more details.

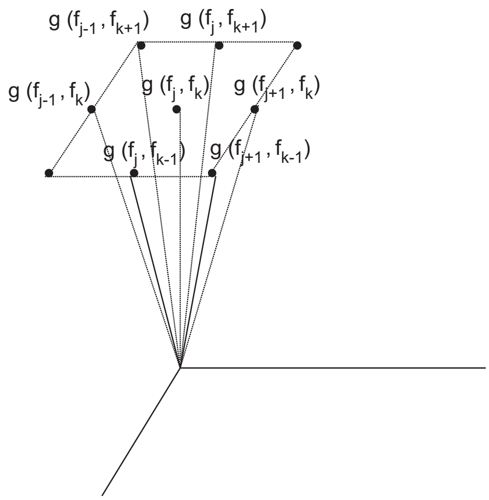

Fig. 7. The approximation for $\partial^2\bar{g}_i/\partial f_{ij}\partial f_{ik}$ is computed from seven values of the function g_i indicated in the diagram; the time index i has suppressed in figure for greater clarity.

derivative, using $\delta^i_{\pm} g(f_i) = g(f_i \pm \delta)/\delta$, yields the following discretization:

$$\begin{aligned}\frac{\partial^2 \bar{g}_i}{\partial f_{i,j}\partial f_{i,k}} &\equiv \frac{\partial^2 g_i}{\partial f_{i,j}\partial f_{i,k}}\bigg|_{f_{i,j}=\bar{f}_{i+1,j}} = \frac{1}{2}(\delta^i_-\delta^j_+ + \delta^i_+\delta^j_-)g_i\bigg|_{f_{i,j}=\bar{f}_{i+1,j}} \\ &= \frac{1}{2\delta^2}[g_i(\bar{f}_{i+1,j}+\delta,\bar{f}_{i+1,k}) - 2g_i(\bar{f}_{i+1,j},\bar{f}_{i+1,k}) - g_i(\bar{f}_{i+1,j}+\delta,\bar{f}_{i+1,k}-\delta) \\ &\quad + g(\bar{f}_{i+1,j},\bar{f}_{i+1,k}-\delta) + g(\bar{f}_{i+1,j},\bar{f}_{i+1,k}+\delta) - g_i(\bar{f}_{i+1,j}-\delta,\bar{f}_{i+1,k}+\delta) \\ &\quad + g_i(\bar{f}_{i+1,j}-\delta,\bar{f}_{i+1,k})]. \end{aligned} \tag{44}$$

The result in Eq. (44) above has a very significant feature. To evaluate $\partial^2\bar{g}_i/\partial f_{ij}\partial f_{ik}$ one needs to know the values of g_i at the points $f_{ij} = \bar{f}_{i+1,j}$, for all $f_{ij}, j = 1, 2, \ldots, N+i$. Moreover, as required by Eq. (44), the forward rate tree at time t_{i+1} must also contain the following three points, namely $\bar{f}_{i+1,j}, \bar{f}_{i+1,j} \pm \delta$. This feature of the recursion equation is a reflection of a similar property for the case of the American option for equity as in Eq. (72) and shown in Fig. 24.

Fig. 7 is a graphical representation of the seven terms required for the computation of $\partial^2\bar{g}_i/\partial f_{i,j}\partial f_{i,k}$.

The price of the European option for caplets and coupon bonds can be obtained by repeating the backward recursion up to the present time. In Sections 7 and 8 the numerical price of the European option will be compared with those computed from the closed form result in Eqs. (13) and (20) to assess the accuracy of the algorithm. For American option, one needs to perform, for *each* step up to present time, a comparison of the trial option price with the payoff as given in Eqs. (25) and (27).

5. Tree structure of forward interest rates

To minimize the computational complexity and the time of execution, it is mandatory to limit as far as possible the number of possible forward rates for which the option price is computed.

The forward bond measure for the forward interest rates is chosen for Libor instruments and is defined only for future Libor time lattice. The numeraire used for discounting all financial instruments is equal to $B(t, T_n)$ for $T_n \leqslant x < T_{n+\ell}$. The numeraire makes all forward bond prices, defined on the Libor future time lattice, into martingales [7]; the definition of the martingale for this numeraire, together with the forward drift, yields the following result:

$$\mathrm{e}^{-\int_{T_n}^{T_n+\ell} f(t_0,x)} \equiv F(t_0,T_n,T_{n+\ell}) = E_F[F(t_*,T_n,T_{n+\ell})]$$
$$\Rightarrow \alpha(t,x) = \sigma(t,x)\int_{T_n}^{x} \mathrm{d}x' D(x,x';t)\sigma(t,x'), \quad T_n \leqslant x < T_n + \ell.$$

For the American option, future time is discretized as $x = na$ and Libor interval $\ell = a$; hence $T_n = n\ell = na$ lies on the future time lattice with $x_n = T_n$. This in turn, from above equation, yields

$$\alpha(t,x_n) = 0, \quad x_n = T_n \tag{45}$$

and there is no drift for the lattice on which the American option is being computed. In fact, this simplification is the main reason for taking $\ell = a$.

The option price is fixed by, among other parameters, the initial forward interest rate curve $f(t_0,x)$. The option price for only those intermediate (virtual) values of the forward rates need to be considered that contribute to the final option price. For a given initial forward interest rates curve, what this means is that the option price needs to evaluated only for those values of the forward rates that lie on a *tree* (also called a *grid*).

In order to ascertain the forward interest rates grid, it is necessary to start from the initial forward interest rate curve, which from Eq. (29) is given by $f(t_0,x) \to f_{M+1,n}$, where $n = 1,2,\ldots,N+M+1$. Similar to the case for the American equity option discussed in the Appendix, *each* initial value of the forward rates generates an independent tree. Recall that, starting from the initial forward rates $f_{M+1,n}$, to reach the forward interest rates at calendar time m, with forward rates f_{mn}, one needs to take $M+1-m$ steps. Hence the structure of the forward rate tree is given by

$$f^k_{mn} \doteq f_{M+1,n} + k\delta, \quad -(M+1-m) \leqslant k \leqslant +(M+1-m). \tag{46}$$

At lattice time m the forward rate tree has $2(M+1-m)-1 = 2(M-m)+1$ number of values for f_{mn}, centered on the initial forward rate $f_{M+1,n}$. The spacing of the tree is taken to have a *fixed* value δ, which is of $O(s)$. A fixed value of δ is required to obtain a recombining tree; conversely, if δ is taken to vary with time, one gets a dense tree with exponentially more points than the recombining tree [4].

The same result as given in Eq. (46) is obtained if one recurses backwards from the payoff at calendar time t_* to the initial forward curve at t_0. The reason being that for zero drift, that is $\alpha(t,x) = 0$, one has from Eq. (44) that the values of the function g_i at with forward rates on the grid $f_{i,j} = f_{i+1,j}, f_{i+1,j} \pm \delta$ are required to obtain the value of g_{i+1}; if one recurses backwards $M+1-m$ times, one hits the initial forward rate curve and in effect obtains the result given in Eq. (46).

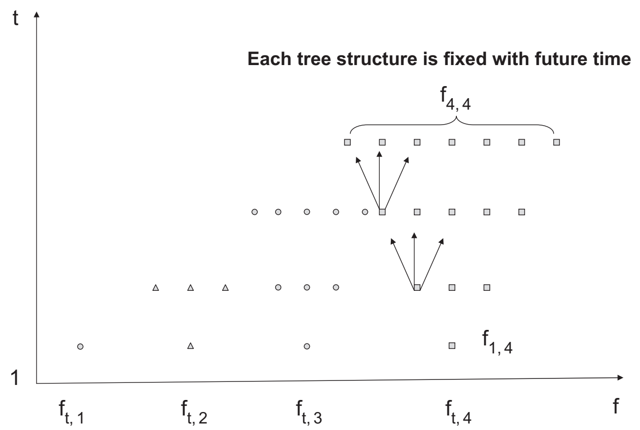

Fig. 8. Forward rates tree structures. Each symbol at bottom (a hollow circle, square and triangle) represents a forward rate f_{mn} at present time. Each initial value of the forward rates evolves as independent tree, with rates evolving from the same initial rate shown with the same symbol. Note for the tree of forward rates, those f_{mn} that emanate from different initial forward rates are independent of each other, and this is indicated by different symbols.

The forward interest rate tree is illustrated in Fig. 8. Compared with Fig. 6, the forward rate at a lattice point (with fixed time t and future time x) has been expanded into a tree structure in a direction orthogonal to the xt lattice. The full forward interest rates simultaneously have infinitely many tree structures and all these tree structures are correlated by the action S given in Eq. (32).

Thus, as one recurses backwards from the payoff function, at any intermediate time the price of the option needs to determined for the forward interest rates only on the grid points. From (25), (27) and (44), the initial trial values $g_I(i)$ that one needs are the option values from the previous step, with the values of the forward rates taken only from the tree structure for lattice time t_i.

6. Numerical algorithm

Given the initial forward interest rates curve, the tree structure from Section 5 can be formed and yields the grid of the forward interest rates up to the expiration date of the option. To get the option price today, one starts from expiration time t_* and evolves backwards in calendar time. All computations are carried out using the lattice theory given in Eq. (35).[5]

Lattice points are labelled by i,j with i labelling calendar time and j labelling future time. Maturity time t_* is labelled by t_1, that is $i = 1$, as shown in Fig. 6. The scaled payoff function g_1 is, in general, a function of all forward interest rates f_{1j} with future time taking $N+1$ values, as shown in Fig. 6, being labelled as $j = 1, 2, \ldots, N+1$. Thus, g_1, which depends on N forward rates, can be represented as $g(f_{11}, \ldots, f_{1i}, \ldots f_{1N+1})$.

Since the step size ε is fixed, the total number of steps $M = t_*/\varepsilon$ for the evolutions backwards in time is consequently also fixed. At time t_i the number of tree points for each forward interest rate is given by $2(M-(i-1))+1 = 2(M-i)+3$; there are $N+i$ number of independent forward rates. Since each forward rate tree has $2(M-i)+3$ points, this leads to the total number of points of the tree at time i being given by $(2(M-i)+3)^{N+i}$. The tree is organized into a multidimensional array,[6] namely $g[2(M-i)+3]_{N+i}$, which is a $N+i$ dimensional array with each index running from $1, 2, \ldots, 2(M-i)+3$.

The size of the multidimensional array for realistic cases can be very big, requiring a large amount of computer memory and leading to codes for the American option that are inefficient. In order to develop an efficient algorithm the multidimensional array is mapped into a *vector array* with a length of $(2(M-i)+3)^{N+i}$. The multidimensional matrix representation of the tree has some advantages since the index of each forward interest rates is presented explicitly. Hence in the recursive steps required for evaluating Eqs. (25), (27) and (44), the matrix representation is the most transparent way of keeping track of the grid points from the previous step that are required for deriving the trial option price for the present step.

To go from the matrix representation to the vector array one needs an algorithm for mapping the indices of the matrix to the index of the array; in particular the multidimensional matrix array $g[j_{N+i}]\ldots[j_p]\ldots[j_2][j_1]$ needs to be mapped into a vector array $g[j]$. For time i let the matrix indices $[j_{N+i}]\ldots[j_p]\ldots[j_2][j_1]$ be assigned specific numerical values; the corresponding vector index j is given by the following mapping:

$$g[j_{N+i}]\cdots[j_p]\cdots[j_2][j_1] = g[j],$$

$$[j_{N+i}]\cdots[j_p]\cdots[j_2][j_1] \to j = \sum_{p=1}^{N+i}(j_p - 1)[2(M-(i-1))+1]^{p-1}. \tag{47}$$

One should note that the matrix representation is never used in writing the codes for this algorithm. The vector array is used for avoiding the use of the multidimensional matrix; only the indices of the matrix are needed as intermediate step to address grid points in the recursion process.

To find the grid points, in particular those that are nearest neighbor and next nearest neighbor as required in evaluating Eq. (44) one needs the mapping in the reverse direction. The mapping from the vector array index to matrix indices is given by the following. The notation used is that the vector index j is recursively updated to $j^{(1)}, j^{(2)}, \ldots, j^{(p)}, \ldots, j^{(N+i)}$; recall the notation $j_1, j_2, \ldots, j_p, \ldots, j_{N+i}$ are the indices labelling the multidimensional matrix. The inverse mapping is a composed of a two-step algorithm, with the matrix index j_p being

[5]Note in this Section $t_0 = 0$.

[6]$g[M]_N \equiv g\underbrace{[M][M]\ldots[M]}_{N}$.

determined and the vector index j being updated to $j^{(p)}$. More precisely, the following is the mapping:

$$g[j] \to g[j_{N+i}] \dots [j_p] \dots [j_2][j_1],$$

where, using modular arithmetic yields

$$\begin{cases} j_{N+i} = \text{Integer}[(j-1)/(2(M-i)+3)^{N+i-1}] + 1, \\ j^{(1)} = j - \text{Integer}[j/(2(M-i)+3)^{N+i-1}], \end{cases}$$

$$\vdots$$

$$\begin{cases} j_p = \text{Integer}[(j^{(p-1)}-1)/(2(M-i)+3)^{p-1}] + 1, \\ j^{(p)} = j^{(p-1)} - \text{Integer}[j^{(p-1)}/(2(M-i)+3)^{p-1}], \end{cases}$$

$$\vdots$$

$$\begin{cases} j_1 = j^{(N+i-1)}, \\ j^{(N+i)} = j^{(N+i-1)} - j^{(N+i-1)} = 0. \end{cases} \tag{48}$$

Note the inverse mapping stops after $N+i$ steps, as indeed it must as this is the total number of forward interest rates at calendar time i. The inverse map returns all the $N+i$ indices j_p of the matrix representation $g[j_{N+i}]\dots[j_2][j_1]$ from the vector index j of the vector array $g[j]$.

In summary, at step i, the final values of option prices are evaluated and stored in a vector array $g_i[(2(M-i)+3)^{N+i}]$. Evolving one step back from t_i to t_{i+1}, in order to evaluate the trial value of $g_{i+1}[j]$, one needs to first map the vector index j to matrix indices $j_p, p = 1, 2, \dots, N+i$ using Eq. (48). The matrix indices are needed for tracking those option prices at step i that are required for deriving $g_{i+1}[j]$ on the grid points. Then, one reverts back to the vector array index from these matrix indices by Eq. (47), and furthermore, obtain the corresponding option values at step i. The recursion process in (25), (27) and (44) is then performed to obtain the trial value of $g_{i+1}[j]$.

Completing one recursion step results in trial values $g_I[(2(M-i)+3]_{N+i}$ for the $(i+1)$th step. The grid points are *dynamic* in nature since one more forward interest rate, namely $f_{i+1,N+i+1}$ has to be *added* at $(i+1)$th step, as shown in Fig. 6. The dimension of the matrix of trial values has to be increased from $N+i$ to $N+i+1$ dimensions, that is $g[j_{N+i}]\dots[j_1] \to g[j_{N+i+1}][j_{N+i}]\dots[j_1]$. The new forward interest rate does not directly influence the option values, but only through the scaling function $B(t_{i+1}, T)$.

The expanded matrix has to be assigned numerical values for the new index j_{N+i+1} in the range of 1 to $2(M-((i+1)-1))+1 = 2(M-i)+1$. The way this is done is to make the values of the expanded matrix *independent* of the new forward rate; in other words, the following assignment is made for the initial trial option price:

$$g[j_{N+i+1}][j_{N+i}]\dots[j_1] \equiv g[j_{N+i}]\dots[j_1], \quad 1 \leqslant j_{N+i+1} \leqslant (2(M-i)+1). \tag{49}$$

For vector array representation, j now takes additional values from $[2(M-i)+1]^{N+i}+1$ to $[2(M-i)+1]^{N+i+1}$. The option values of the vector array for the new values of j, similar to Eq. (49), are made independent of the new value of j; hence

$$g[j] \equiv g[j-[2(M-i)+1]^{N+i)}], \quad [2(M-i)+1]^{N+i}+1 \leqslant j \leqslant [2(M-i)+1]^{N+i+1}. \tag{50}$$

The mapping in Eq. (50), in going from the left-hand to the right-hand side of the equation *shifts*, by a constant, the argument of the new index j and thus bringing it back into the old range; as j runs through the (additional) new values, the expanded vector takes values from the old array that are a constant shift from the new values of j; it can be seen by inspection that Eq. (50) assigns values to the expanded vector array consistent with the labelling of the new matrix elements given in Eq. (49).

The trial option value is compared with the payoff value at $(i+1)$th step. The higher value is retained at the $(i+1)$th step as the final value of $g[j_{N+i+1}][j_{N+i}]\dots[j_1]$, which is the American option at the grid points.

The price of the American option at present is obtained by repeating the recursion until $i = M + 1$. The European option price can also be derived following the same algorithm, but without making the comparison with payoff value at each step of the recursion.[7]

To initiate the numerical algorithm the initial forward rates curve and all the parameters in the lattice Lagrangian have to be specified. The numerical algorithm is given as follows:

- Input initial forward rates curve and parameters.
- Generate the payoff for maturity time $i = 1$ and store in (both) the vector arrays $Geuro_{old}$ and $Gamerican_{old}$ for European and American option.
- For $i = 2$ to $M + 1$
  1. Recurse one step back from $Geuro_{old}$ and $Gamerican_{old}$ to get trial values $Geuro_{new}$ and $Gamerican_{new}$ using Eqs. (25), (27) and (44).
  2. Expand $Geuro_{new}$ and $Gamerican_{new}$ from $N + i - 1$ to $N + i$ dimension so as to address the dynamics of grid using Eq. (50).
  3. Compute the payoff value at step i without discounting and store result in $Gamerican_{old}$.
  4. Compare $Gamerican_{old}$ with $Gamerican_{new}$ and store the larger one in $Gamerican_{old}$. Replace values in $Geuro_{old}$ with values in $Geuro_{new}$.
  5. End for.

The vector array has to be used to assign option values for each point in the forward rate grid. The length of the vector array can be very large, and frequently addressing the elements of the array may cause problem of over the stack or even giving wrong values. However, programming languages have a feature of dynamically addressing the location of the vector array, which helps to avoid these problems.

Note that the first order of the recursion contributes significantly to the final value. Furthermore, the drift in forward bond measure is zero at 3 monthly lattice of points. The tree structure has to be enough wide to include information about the changes in the value of the forward rates. One has the freedom of increasing the width of the tree by setting the prefactor of $\delta = O(s)$. Since σ that is being used is the volatility for a one day change of forward interest rate, to obtain s the real days in each step must be multiplied into it.

The above numerical algorithm yields only one value for both American option and European option; in order to get values for a time series or values depending on different values of the various parameters, the entire algorithm needs to be repeated.

7. Numerical results for caplets

A caplet on Libor and maturing when the caplet becomes operational was analyzed. The initial forward interest rate curve as well as the volatility function was taken from the Libor market; the propagator $\tilde{D}_{i,jk}$ is assigned numerical values taken from caplet data [8].[8] For simplicity, take the time lattice $\varepsilon = 3$ months. The present time for the caplet is taken to be from 12th September 2003 to 7th May 2004, with maturity at fixed time in the 12th December 2004. For early exercise the American option on the caplet can expire at *five* fixed times.

For M time steps and $N + 1$ forward rates in first step, at step i there are $Q = (2(M - i) + 3)^{N+i}$ option prices that need to be determined. The total number of option prices for the whole algorithm is $\mathcal{O} = \sum_{i=1}^{M+1}(2(M - i) + 3)^{N+i}$. Thus, for a caplet at 12th September 2003, $N = 0$ and $M = 5$ (since $M = t_*/\varepsilon$ and $\varepsilon = 3$ months), the number of option prices that need to be determined is $\mathcal{O} = 1,304$. The total number of option prices $\mathcal{O}$ increases rapidly with increasing M, with $\mathcal{O} = 14,758,719$ for $M = 10$ (Fig. 9).

The caplet tree of the (relevant) forward rates is built with $\delta = 2s$; all computations are carried out only for the values of the forward rates taking values in the tree. Caplet volatility is taken from the market by moving average on the historical data, and at 12th September 2003 is given in Fig. 25 [8].

[7]The European option price is always evaluated (at the same time as the American option) for carrying out consistency checks on the numerical results.

[8]Market data was used for pricing caplets to demonstrate the flexibility of the numerical algorithm.

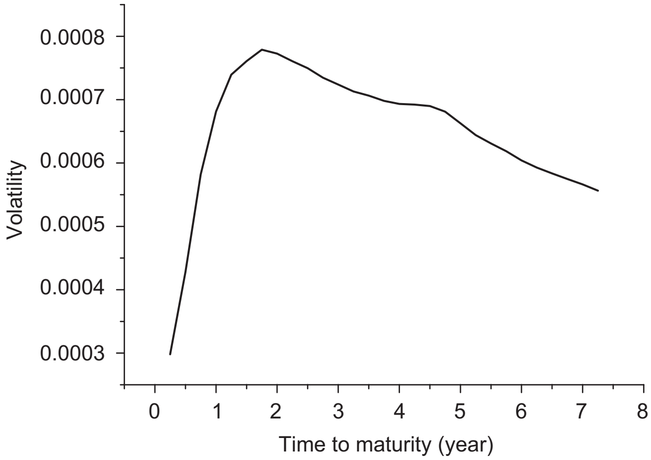

Fig. 9. Forward interest rates volatility $\sigma(\theta)$, with dimension year^{-1}, on 12th September 2003 versus maturity of time. $\sigma(\theta)^2 = \sqrt{\langle \delta f^2(t,\theta)\rangle_c}$; the average $\langle \ldots \rangle$ is obtained by an averaging over 60 days of historical Libor data.

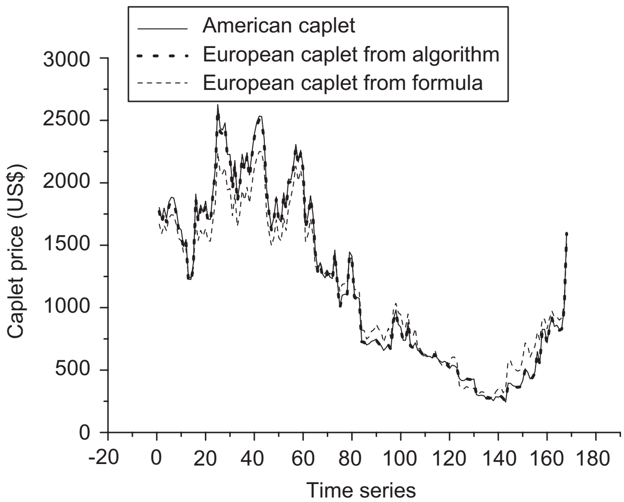

Fig. 10. American and European caplet prices for fixed maturity at 12.12.2004 versus time t_0 (12.9.2003–7.5.2004). (European caplet from formula and algorithm). The normalized root mean square error for the European caplet price between the numerical value and formula is 7.3%.

In Ref. [8], the daily price from 12th September 2003 to 7th May 2004 of the option on Eurodollar futures contracts expiring 13 December 2004 with a strike price 98 were computed. It was shown the field theory pricing formula is fairly accurate; the same instrument is studied numerically using the lattice field theory of interest rates.

Both European and American options are computed and the European results are used to check the accuracy of the algorithm by comparing the numerical results with results from closed form pricing formula for European options.

The American option can be exercised at any time before it's expiry day, which means one should set ε to be very small and N to be very big; doing so would require a huge memory and very long time to run the program since the possible option values Q for each step would then be a large number.

In Fig. 10 the numerical results of caplet are shown, and it is seen the numerical results are quite accurate even for a large value of $\varepsilon = 3$ months; for this value of ε this program can generate 167 daily prices by running for less than 2 s on a desktop.

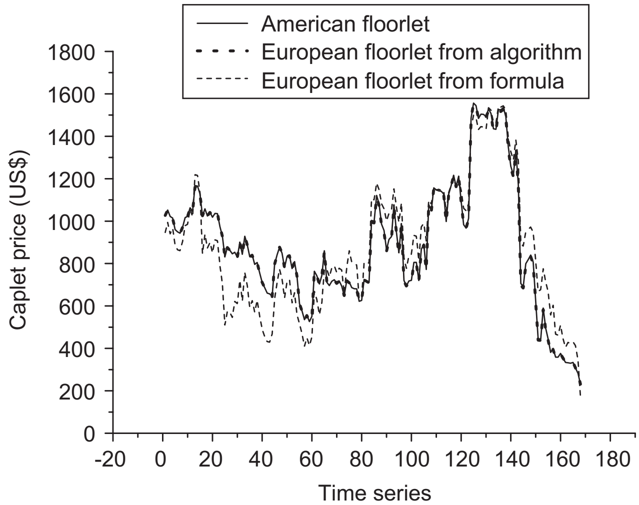

Fig. 11. American and European floorlet prices for fixed maturity at 12.12.2004 versus time t_0 (12.9.2003–7.5.2004), (European floorlet from formula and algorithm). The normalized root mean square error of the European floorlet price from numerical algorithm and formula for floorlet is 8.8%.

Floorlet prices result is shown in Fig. 11.

Besides accuracy, the numerical results need to be consistent with the general properties of the various options. In particular [4], the American caplet (put option) must always be more expensive than European caplet since the American option includes the European option as a special case; however, American floorlet (call option), in the absence of a dividend, is always equal to European floorlet. The normalized difference between American and European options is shown in Fig. 12.[9]

The results are seen to be consistent with the general properties of the American and European options; the normalized difference between American and European caplet is strictly positive showing that the American caplet is always more expensive than the European caplet; on the other hand, the gap between American and European floorlet can have negative values, showing that, within the accuracy of the numerical algorithm, their difference is zero.

Although the interval between evolving steps ε is set equal to 3 months, one can always decrease this interval to get more accurate results. The American option is more expensive on decreasing the interval ε since one needs to pay more to have an option that can be exercised on more occasions before the expiry date. One can consider the American option being exercised at fewer instants of time as Bermudan options. A *Bermudan option* can be exercised at a number of pre-fixed times and is equal to a basket of European options, with the difference that once the Bermudan option is exercised all the remaining European options become invalid. A Bermudan option is always cheaper than an American option but more expensive than European option. Some numerical results for the European, Bermudan and American caplets are shown in Fig. 13, and are seen to be consistent with the general requirement for these options.

Put–call parity for the European caplet and floorlet is given by [7,8]

$$Caplet(t_0, t_*) - Floorlet(t_0, t_*) = \ell VB(t_0, t_* + \ell)[L(t_0, t_*) - K]. \tag{51}$$

The third argument T, indicating when the caplet becomes operational, is suppressed since the numerical algorithm only studies the price of a caplet and floorlet for $t_* = T$. The result in Fig. 14 verifies that put–call parity is valid for the European option prices generated by the numerical algorithm.

[9]One expects that by decreasing the time lattice size ε, the difference between American caplet and European caplet should be enlarged and become more significant compared with the error (deviating from zero) of the difference between American floorlet and European floorlet.

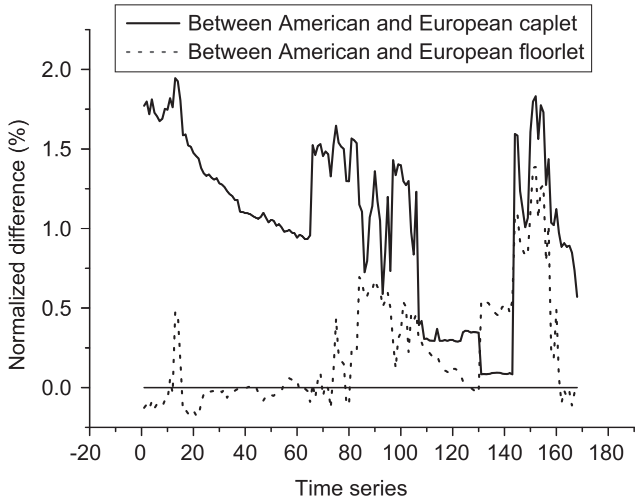

Fig. 12. The (%) normalized difference between the American and European options for both caplet and floorlet versus time t_0 (12.9.2003–7.5.2004). Within the numerical accuracy of the computation, the European and American floorlet prices are equal, whereas the caplet prices for the European case is always less than the American case, as expected.

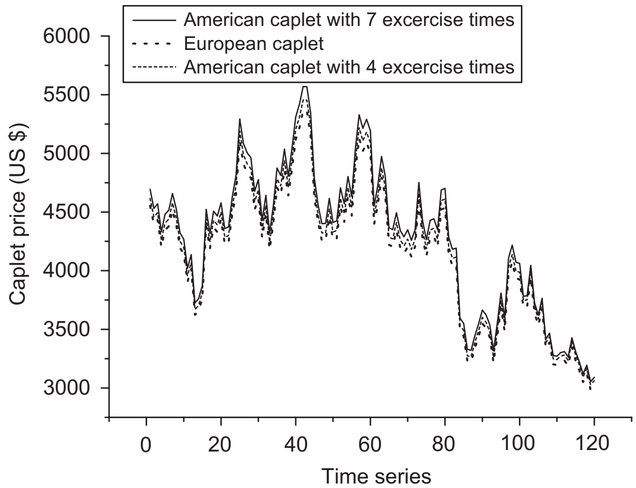

Fig. 13. Caplet prices versus time t_0 for the European and American options, with four and seven possible exercise times.

8. Numerical results for coupon bond options

For the coupon bond option case, the forward rates tree is built with the value of $\delta = 6s$. The main focus of the study of the American coupon bond option is not empirical, but instead is to develop an efficient and accurate algorithm. Given the complexity of the instrument, a model is assumed for the volatility and the initial value of the forward rates and the numerical study analyzes the accuracy of the algorithm for the model. No market data is used for studying the American coupon bond option price.

The initial lattice forward interest rates is taken as below

$$f_{mn} = f_0(1 - e^{-\lambda(n-m)}), \tag{52}$$

where f_0 is a prefactor used to get the same magnitude as the real market forward rates; let $f_0 = 0.1$ and choose $\lambda = 1$ so that f_{mn} is of the order of 10^{-2}.

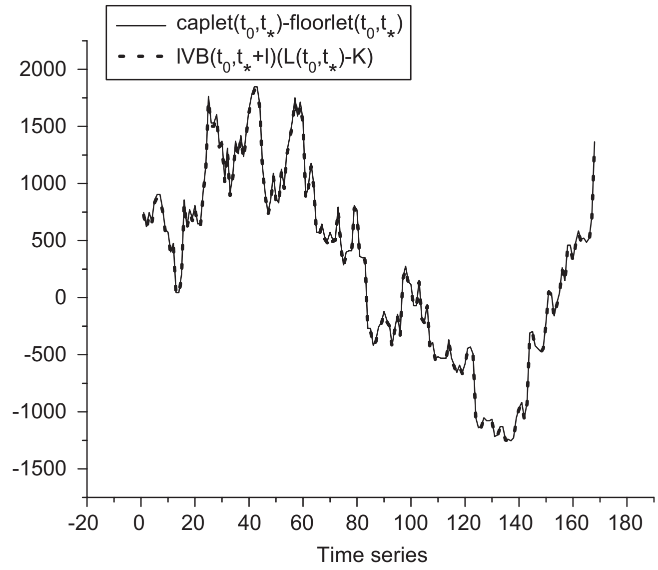

Fig. 14. Put–call parity for the European caplet and floorlet versus time t_0 (12.9.2003–7.5.2004). The normalized root mean square error is 3.2%.

Following Bouchaud and Matacz [11], volatility is taken to have the following form (the parameters are fixed by historical forward interest rates data):

$$\sigma(\theta) = 0.00055 - 0.00026\exp(-0.71826(\theta - \theta_{min})) + 0.0006(\theta - \theta_{min})\exp(-0.71826(\theta - \theta_{min})), \quad \theta = x - t, \tag{53}$$

where $\theta_{min} = 3$ months.

The volatility, computed as volatility of daily change in the forward rates, is given from historical data by $\sigma^2(t,\theta) = \langle \delta f^2(t,\theta)\rangle_c$, $\delta f(t,\theta) = f(t+1,\theta) - f(t,\theta)$. Thus, in building the tree where each step ε is 3 months, we have to multiply the actual days of 3 months into $\sigma(\theta)$ to obtain the dimensionless volatility s_{mn}; from the definitions of the lattice variables $\theta = x - t \to (N - n + m)\varepsilon$. Since the number of trading days in 3 months is 65 and $\varepsilon = a = \frac{3}{12} = 0.25$ years, the dimensionless volatility for the case of the coupon bond option is given by ($\theta_{min} = \varepsilon$)

$$\begin{aligned} s_{mn} &= 65\sqrt{0.25}\sqrt{0.25}\sigma(\sqrt{0.25}(N - n + m)) \\ &= 16.25[0.00055 - 0.00026\exp(-0.35913(N - n + m - 1) \\ &\quad + 0.0003(N - n + m - 1)\exp(-0.35913(N - n + m - 1))]. \end{aligned} \tag{54}$$

The stiff propagator is given by Baaquie and Bouchaud [6], with parameters taken to have the following values $\tilde{\lambda} = 1.790$/year; $\tilde{\mu} = 0.403$/year; $\eta = 0.34$.

The numerical study considers the coupon bond option $c_1B(t_*, 1/4) + c_2B(t_*, 1/2)$ that matures in one years time, that is $t_* - t_0 = t_* = 1$ year and has a duration of six months, with two coupon payments and each is paid every three months; the fixed coupon rate is taken to be equal to c and the principal amount equal to 1. Thus, the payoff function at time $t_* = 1$ year for the put option is given by

$$S(t_*) = \left(K - \sum_{i=1}^{2} c_i B(t_*, T_i)\right)_+, \tag{55}$$

where $T_1 = 1.25$ year, $T_2 = 1.5$ year, $c_1 = c$ and $c_2 = c + 1$. Note taking the $c = 0$ limit converts the coupon bond into a zero coupon bond.

Table 1
The correlators G_{ij} between different forward bond prices

G_{ij}	$i = 1$	$i = 2$
$j = 1$	1.669×10^{-8}	3.624×10^{-8}
$j = 2$	3.624×10^{-8}	7.924×10^{-8}

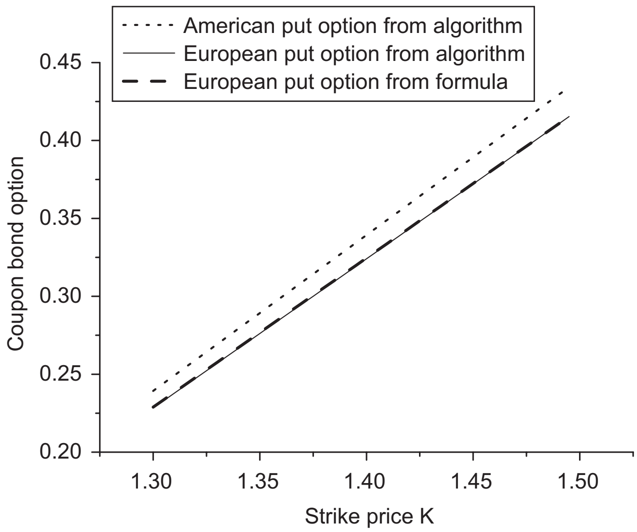

Fig. 15. Prices of coupon bond American and European put option that matures in one year and with a duration of six months, during which two coupons at the rate of $c = 0.05$ are paid every three months versus strike price K (European option from formula and algorithm). The normalized root mean square error between the numerical value and formula for the European option is 0.17%.

For this coupon bond option, $N = 1$ and set $M = 4$; this yields the total number of option prices to be evaluated equal to $\mathcal{O} = 1,293$. This number increases more rapidly than for the caplet case, and for $M = 10$ it reaches $\mathcal{O} = 118,507,277$.

Comparing the numerical value of the European coupon bond option with the approximate formula given in Eq. (20) provides a check on the numerical result. The approximate formula is an expansion in the volatility of the forward rates s_{mn}, and as long as this volatility is small, the numerical and approximate results should agree.

For the specific case that is being studied numerically the coefficient A in Eq. (21) that appears in European coupon bond option given in Eq. (20) has the closed form expression given by

$$A = \sum_{ij=1}^{2} J_i J_j \left[G_{ij} + \frac{1}{2} G_{ij}^2 \right] + O(G_{ij}^3). \tag{56}$$

Note and $J_i = c_i F_i$, with $F_1 = 0.982321$ and $F_2 = 0.963426$.

The numerical values for G_{ij}, the correlator of the forward bond prices, are given in Table 1.

Numerical results for coupon bond option prices with changing strike price K and coupon rate c are given in Figs. 15 and 16. The numerical value of the European coupon bond option is seen to be approximately equal to the closed form approximate formula in Eq. (20); as required by consistency, the American put option always has a higher price than European put option.

For completeness, the special case of the American option on a zero coupon bond, obtained by setting $c = 0$ in Eq. (55), is given in Fig. 17 and shows all the features required by the consistency of the option prices.

The algorithm has been checked for internal consistency by plotting the prices of the coupon bond American put options, European put options and the payoff function against the value of the coupon bond

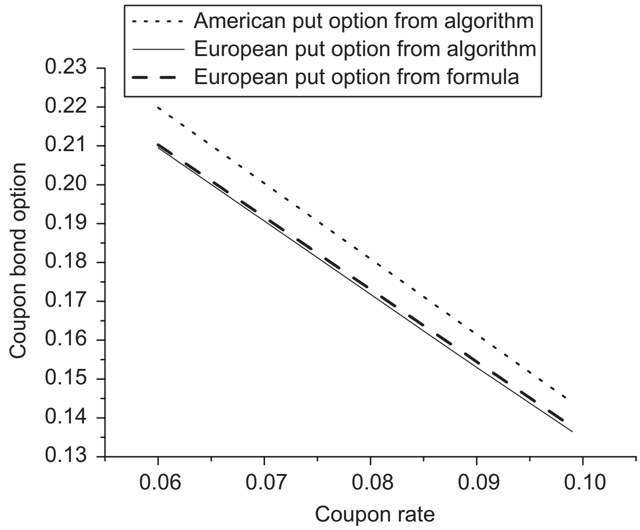

Fig. 16. Prices of coupon bond American and European put option that matures in one year and with a duration of six months, during which two coupons at the rate of $c =$ are paid every three months with strike price $K = 1.3$ versus coupon rate c (European option from formula and algorithm) are shown. The normalized root mean square error between the numerical result and formula for European option is 0.73%.

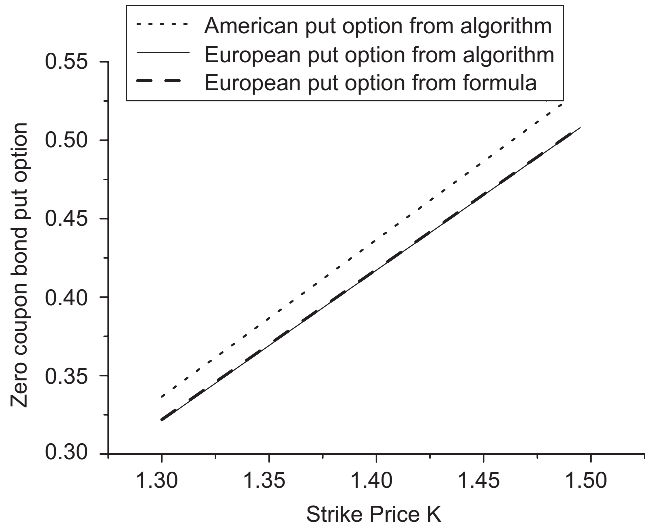

Fig. 17. Prices of American and European put options for *zero coupon bond* with half year duration and one year maturity versus strike price K (European options from formula and algorithm). The normalized root mean square error between the numerical value and formula for the European option is 0.16%.

$B(t_0)$, which were generated by varying the coupon rate c, where $B(t_0)$ is value of the coupon bond at t_0 and is given by

$$B(t_0) \equiv \sum_{i=1}^{N} c_i B(t_0, T_i). \tag{57}$$

Fig. 18 shows that the results are consistent with the general properties of these options, with the price of the American option always being higher than the European option, and the American option joins the payoff

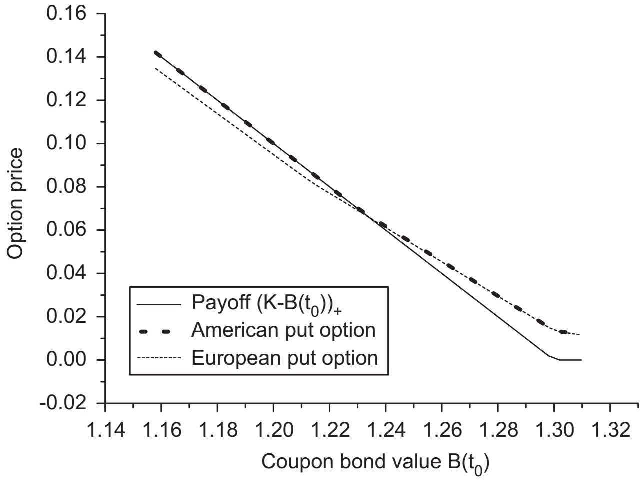

Fig. 18. The coupon bond payoff function and the American and European coupon bond put option prices versus the underlying coupon bond value $B(t_0) = \sum_{i=1}^{2} c_i B(t_0, T_i)$.

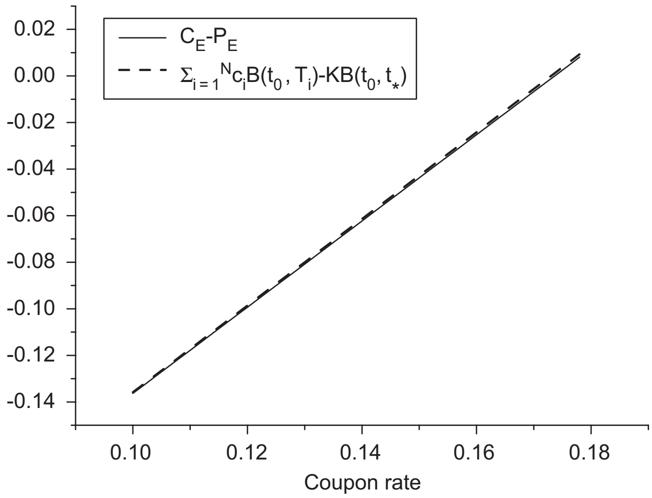

Fig. 19. Graph of put–call parity $C_E(t_0, t_*, K) - P_E(t_0, t_*, K) = \sum_{i=1}^{N} c_i B(t_0, T_i) - KB(t_0, t_*)$ for numerical prices of the European coupon bond options versus coupon rate. The normalized root mean square error for put–call parity is 1.53%.

with the same slope as the payoff function for small coupon bond value $B(t_0)$, where for large values of $B(t_0)$ the American options joints the European option, as discussed in Ref. [1].

Another check of the algorithm is the put–call parity for coupon bond option. European options obey the equation [9]

$$C_E(t_0, t_*, K) - P_E(t_0, t_*, K) = \sum_{i=1}^{N} c_i B(t_0, T_i) - KB(t_0, t_*). \tag{58}$$

The numerical results in Fig. 19 shows the accuracy of the algorithm and the normalized root mean square error is only 1.53%.

9. Put–call inequalities for American coupon bond option

In analogy with Eq. (58) and the inequalities for the put and call American options for equity given in Eq. (74), one can consider the following inequalities for the case of American options on coupon bonds:

$$B(t_0) - K \leqslant C(t_0, t_*, K) - P(t_0, t_*, K) \leqslant B(t_0) - B(t_0, t_*)K: \text{ Incorrect.} \tag{59}$$

On graphing the three expressions in the above equation, as shown in Fig. 20, it is seen that the put–call inequalities are *incorrect*.

Instead of the above incorrect inequality, a *conjecture* is made that the American coupon bond options satisfy the following modified inequalities. The coupon bond value $B(t_0)$ at time t_0 in Eq. (59) is replaced by the *present value* of the payoff function, namely $F(t_0)$—the forward price of the payoff function at time t_0—and which is given by

$$F(t_0) \equiv \sum_{i=1}^{N} c_i F(t_0, t_*, T_i).$$

Hence we have postulate the following inequalities:

$$F(t_0) - K \leqslant C(t_0, t_*, K) - P(t_0, t_*, K) \leqslant F(t_0) - KB(t_0, t_*). \tag{60}$$

On numerically checking this inequality, as shown in Fig. 21, it is seen that the American coupon bond option in fact does satisfy the conjectured inequalities! A straightforward no arbitrage argument can be shown to yield inequalities given in Eq. (60).

In analogy with the conjecture for the inequalities obeyed by American coupon bond options, the following inequalities are conjectured for the American caplet and floorlet:

$$F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - K] \leqslant Caplet(t_0, t_*) - Floorlet(t_0, t_*)$$
$$\leqslant F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K] \tag{61}$$

which can also be expressed as follows:

$$Caplet(t_0, t_*) - Floorlet(t_0, t_*) - F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - K] \geqslant 0,$$

$$F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K] - Caplet(t_0, t_*) - Floorlet(t_0, t_*) \geqslant 0.$$

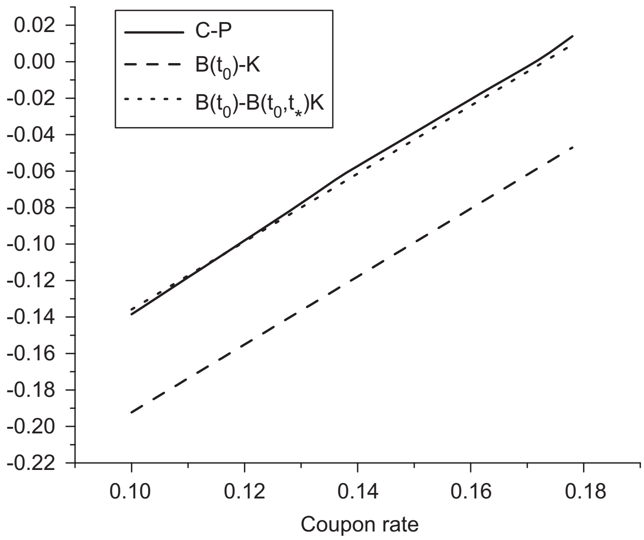

Fig. 20. Numerical result showing the *incorrectness* of put–call inequalities $B(t_0) - K \leqslant C(t_0, t_*, K) - P(t_0, t_*, K) \leqslant B(t_0) - B(t_0, t_*)K$ of American coupon bond option that are analogous to the equity case versus coupon rate.

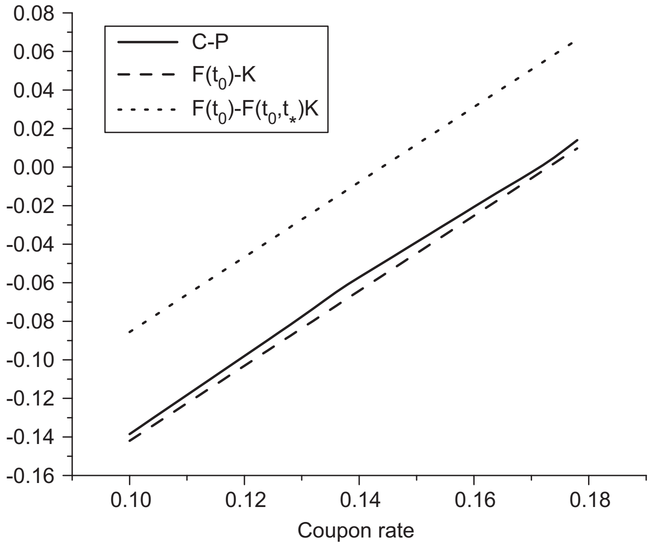

Fig. 21. Numerical result confirming the *conjectured* put–call inequalities of American coupon bond option $F(t_0) - K \leqslant C(t_0, t_*, K) - P(t_0, t_*, K) \leqslant F(t_0) - KB(t_0, t_*)$ versus coupon rate; $F(t_0) \equiv \sum_{i=1}^{N} c_i F(t_0, t_*, T_i)$.

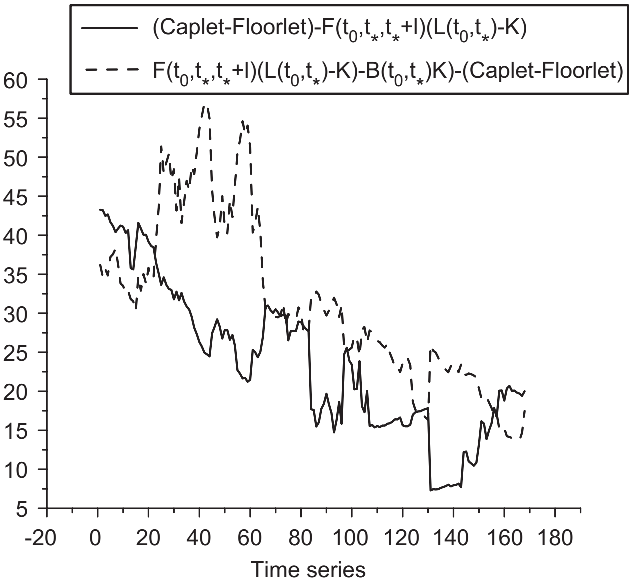

Fig. 22. Put–call inequality, to be obeyed by the price of a caplet American put and call options, requires that $Caplet(t_0, t_*) - Floorlet(t_0, t_*) - F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - K] \geqslant 0$ and $F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K] - Caplet(t_0, t_*) - Floorlet(t_0, t_*) \geqslant 0$. From the diagram one can see that both the expressions are positive, as required.

Fig. 22 shows that the conjectured inequalities indeed do hold for the numerical prices of the American caplet options.

The conjecture for the American caplet and floorlet is not as significant as the one for the American coupon bond option since the numerical prices also satisfy the inequalities that are similar to the equity inequalities in Eq. (74), namely

$$B(t_0, t_* + \ell)[L(t_0, t_*) - K] \leqslant Caplet(t_0, t_*) - Floorlet(t_0, t_*) \leqslant B(t_0, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K]. \quad (62)$$

The numerical results for American caplet and floorlet show that they obey the inequalities given in Fig. 23.

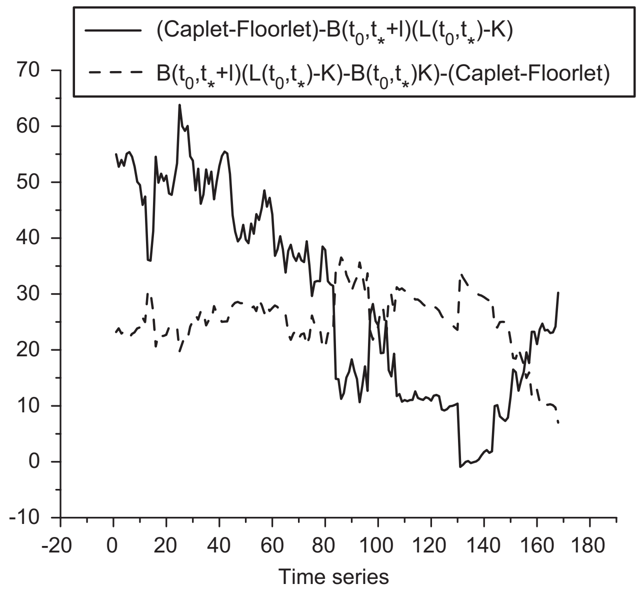

Fig. 23. Put–call inequality obeyed by the price of a caplet American put and call options. $B(t_0, t_* + \ell)[L(t_0, t_*) - K] \leqslant Caplet(t_0, t_*) - Floorlet(t_0, t_*) \leqslant B(t_0, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K]$.

10. Conclusions

An efficient and accurate numerical algorithm has been developed and implemented for pricing American options for interest rate instruments. The procedures that are presently being used [2] are all based on the HJM-model and BGM model and use variants of the binomial tree to build the tree for the interest rates and coupon bonds; the complexity of the tree in the HJM-model and BGM model is determined by how many factors are driving the interest rates.

The approach of quantum finance to American options is radically different. One starts with the pricing kernel, for which a model is written from first principles. The payoff function for the American option is propagated backwards on a time lattice using the pricing kernel, and entails performing a path integral numerically. The numerical path integral can be interpreted as generating the values of the American option on a tree of forward interest rates. At each step the trial American option value obtained is compared with the payoff function.

The (lattice) forward interest rates $f(t,x) \to f_{mn}$ are directly involved in the recursion equation. Any attempt to replace f_{mn} by any collection of white noise, as is the case for the HJM-model and BGM model, makes the whole numerical computation intractable. Furthermore, the non-trivial correlation between the changes in the option price due to the propagator $\tilde{D}_{m,jk}$ is easily incorporated in the recursion equation, as can be seen from Eq. (43); this correlation cannot be incorporated in the numerical procedures based on the HJM-model and BGM model. The prices of the American option for caplets and coupon bonds have been shown to be consistent, obeying all the constraints that necessarily follow from the principles of finance.

The comparison of the numerical values of the European caplet and coupon bond option with the caplet formula and the approximate coupon bond option formula, respectively, showed that the numerical algorithm in general is over 95% accurate, reaching an accuracy of 99% for the coupon bond option. An additional benefit of the comparison is that it provides a proof of the accuracy of the approximate coupon bond option price for low interest rates volatility.

The entire computation has been carried out on a desktop computer and with a small lattice of about 10–20 lattice points, with the option price for each set of parameters requiring only a few seconds of computation time. Even such a crude approximation gives excellent results, showing the possibility of using such algorithms for practical applications.

A conjecture for the put–call inequalities was made for the American coupon bond option and caplet was made based on the analysis of the numerical results. The fact that these inequalities seem to hold quite robustly for the numerical results obtained adds confidence to the correctness of this conjecture. A derivation of the conjecture from the principles of finance would confirm the correctness of the conjecture.

The numerical algorithm developed in this paper opens the way to the study of all varieties of American options, both for interest rate instruments and for correlated equity instruments as well.

Acknowledgments

One of us (BEB) thanks Sanjiv Das for his consistent encouragement and many instructive discussions. We thank Mitch Warachka for carefully reading the manuscript and Jitten Bhanap for useful discussions. Bloomberg Singapore is thanked for generously providing the caplet data.

Appendix A. American option on equity

The Black–Scholes case provides a simple example for understanding the more complicated case. Montagna et al. [5] have given a path integral numerical evaluation of the American option for Black–Scholes equity. In order to understand the American option for the more complex case of interest rates, the derivation of Ref. [5] is briefly reviewed. The notation followed in this Appendix is the same as the one used for analyzing lattice interest rates in Section 4.

The Black–Scholes models the time evolution of asset prices. Consider a time lattice with time running *backwards*, that is, $t_i = (N-i)\varepsilon, i = 0, 1, \ldots, N$, with $t_* = N\varepsilon$ usually being the expiration time for an option. In terms of the logarithm of the asset price $S_i = \mathrm{e}^{z_i}$, with $z_i \equiv z(t_i)$ and discretized velocity $\mathrm{d}z/\mathrm{d}t = (z_i - z_{i+1})/\varepsilon$, the Black–Scholes Lagrangian [3] is given by

$$\mathscr{L}_{BS}(i) = -\frac{1}{2\sigma^2}\left(\frac{z_i - z_{i+1}}{\varepsilon} - \alpha\right)^2 - r, \tag{63}$$

where $\alpha = r - \sigma^2/2$. Let the boundary conditions be given by $z_0 = z; z_N = z'$; the action and the pricing kernel are then given by [3]

$$S_{BS} = \varepsilon\sum_{i=0}^{N-1}\mathscr{L}_{BS}(i) = \sum_{i=0}^{N-1}\mathscr{L}(i),$$

$$\mathscr{L}(i) = \varepsilon\mathscr{L}_{BS}(i) = -\frac{1}{2s^2}(z_i - z_{i+1} - \tilde{\alpha})^2 - \tilde{r},$$

$$p(z', z; N) = \tilde{\mathscr{N}}\prod_{i=0}^{N-1}\int \mathrm{d}z_i\, \mathrm{e}^{S_{BS}},$$

$$p(z', z; 1) = \mathscr{N}\exp\{\mathscr{L}\} = \sqrt{\frac{1}{2\pi s^2}}\exp\left\{-\frac{1}{2s^2}(z - z' - \tilde{\alpha})^2 - \tilde{r}\right\}, \tag{64}$$

with dimensionless parameters $s^2 = \varepsilon\sigma^2, \tilde{\alpha} = \varepsilon\alpha$ and $\tilde{r} = \varepsilon r$.

Consider a European put option P_i maturing at time $N\varepsilon = t_*$ in the future, with maturity time labelled by $i = 0$; the payoff function is given by $(K - \mathrm{e}^z)_+$, where K is the strike price.[10] Since time is running backwards, the pricing kernel gives the price of the European put option at time $i = N$ by the following equation:

$$P_E(z', N) = \int_{-\infty}^{+\infty} \mathrm{d}z p(z', z; N)(K - \mathrm{e}^z)_+. \tag{65}$$

Consider the case of the American put option with possibility of an early exercise. The payoff of the American option is the same as the European option, with the additional freedom that the holder of the option

[10] The price of an American call option, for a non-dividend paying stock, can be shown to be equal to the European call option [1].

can exercise the option anytime from the present to its theoretical maturity date t_*. Since time is divided into short intervals of spacing ε, early exercise of the option can only take place at the discrete time instants of $t_i = i\varepsilon$. To find the price of the American option $P(t)$ one propagates the payoff function backwards in time. At time slice t_i the American option has a price given by $P(t_i)$. To determine the option price at next (earlier) instant t_{i+1} one propagates $P(t_i)$ backwards in time to obtain an initial trial value of the American option at t_{i+1}, called $P_I(t_{i+1})$. The actual value of the American option at t_{i+1} is given by the maximum of the (non-discounted) payoff function and $P_I(t_{i+1})$; that is

$$P(t_{i+1}) = \mathrm{Max}\{P_I(t_{i+1}), (K - \mathrm{e}^{z_{i+1}})_+)\}. \tag{66}$$

In the path integral the pricing kernel is used for computing the initial trial option price $P_I(t_{i+1})$; Eqs. (65) and (64) yield

$$P_I(t_{i+1}, z') = \int_{-\infty}^{+\infty} \mathrm{d}z p(z', z; 1) P(t_i, z) = \mathcal{N} \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{\mathscr{L}(z',z)} P(t_i, z) \tag{67}$$

$$= \mathrm{e}^{-\tilde{r}} \sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} \mathrm{d}z \exp\left\{-\frac{1}{2s^2}(z - z' - \tilde{\alpha})^2\right\} P(t_i, z). \tag{68}$$

Almost all cases of interest have fairly small volatility, that is, $s \simeq 0$; for small s the most efficient procedure for evaluating the integral in Eq. (68) is to Taylor expand the function $P(t_i, z)$ about the very sharp maximum of the Gaussian part of the integrand located at the point $z' + \tilde{\alpha} \equiv \bar{z}$. Denoting differentiation with respect to z by prime yields the Taylors expansion

$$P(t_i, z) = P(t_i, \bar{z}) + (z - \bar{z})P'(t_i, \bar{z}) + \tfrac{1}{2}(z - \bar{z})^2 P''(t_i, \bar{z}) + \cdots. \tag{69}$$

Using the fact that

$$\sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-(1/2s^2)(z-\bar{z})^2} = 1, \quad \sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-(1/2s^2)(z-\bar{z})^2}(z - \bar{z}) = 0,$$

$$\sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-(1/2s^2)(z-\bar{z})^2}(z - \bar{z})^2 = s^2 \tag{70}$$

yields, from Eqs. (68) and (69), the following recursion equation:

$$P_I(t_{i+1}, z') = \mathrm{e}^{-\tilde{r}}[P(t_i, \bar{z}) + \tfrac{1}{2}s^2 P''(t_i, \bar{z})] + O(s^4). \tag{71}$$

Discretizing the values of $\bar{z}$ into a grid of spacing δ of $O(s)$ yields

$$P_I(t_{i+1}, z') \simeq \mathrm{e}^{-\tilde{r}}\left[P(t_i, \bar{z}) + \frac{1}{\delta^2}[P(t_i, \bar{z} + \delta) - 2P(t_i, \bar{z}) + P(t_i, \bar{z} - \delta)]\right]. \tag{72}$$

Note that to obtain the value of $P_I(t_{i+1}, z')$ in Eq. (72) one needs the values of option prices at the earlier time at three distinct points, namely $P(t_i, \bar{z}), P(t_i, \bar{z} \pm \delta)$. By induction, it follows that as one recurses back in time, the number of points at which the option price can be obtained collapses into a single point. Hence, in order to find the option price at a some particular value at present, which in the notation being used is at $t_N = 0$, one needs to create a *tree* of values for z_i, at which points the recursion equation will evaluate the value of the option; the tree is illustrated in Fig. 24.

As shown in Fig. 24, the points on the tree grow linearly with each step in time. The values of z_i on the tree are taken to have a spacing of $\delta = s$ so that the spread of the z_i values on the tree can span the interval required for obtaining an accurate result from the integration. The tree at time t_i has the following values for z_i, namely:

$$z_i^{(k)} \doteq z_N + \tilde{\alpha} + ks, \quad k = -(N - i), \ldots, +(N - i). \tag{73}$$

At a given time t_i, the tree consists of $2(N - i) + 1$ values of z_i, centered on the $S = \mathrm{e}^{z_N}$, namely the value of the stock at initial time for which the price of the American option is being computed.

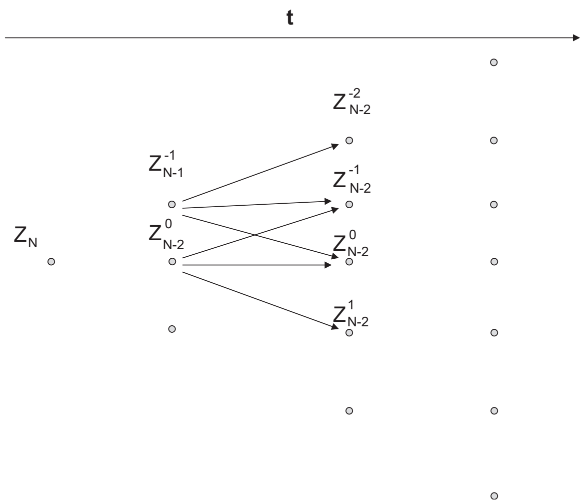

Fig. 24. Tree of stock values, with number of points growing linearly with maturity time.

Table 2
Numerical prices of American and European put options for the parameters $T = 0.5$ year, $r = 0.1$/year, $\sigma = 0.4, K = 10, \varepsilon = T/100$, as a function of the possible present time stock prices S

S	American put	Numerical European put	European from Black–Scholes
6.0	4.00	3.558	3.558
8.0	2.095	1.918	1.918
10.0	0.922	0.870	0.870
12.0	0.362	0.348	0.348
14.0	0.132	0.128	0.128

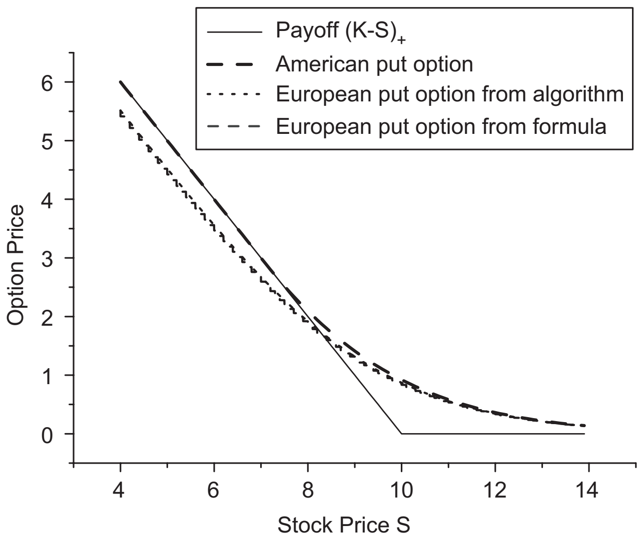

Fig. 25. Price of the American stock option versus stock price S.

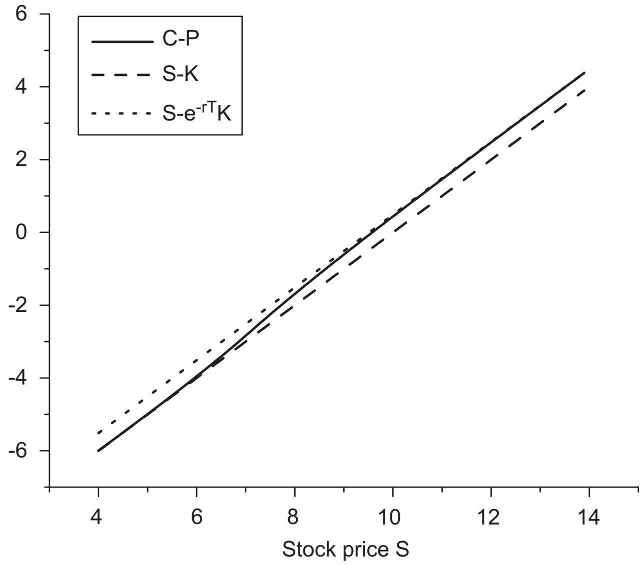

Fig. 26. Put–call inequality for the American stock option.

The algorithm expressed in Eqs. (66) and (72) were numerically tested and yield results (Table 2) that are fairly accurate as well as consistent with those obtained in Ref. [5].

The American and European put option prices, together with the payoff function for the put option, are shown in Fig. 25 and are seen to be consistent with the discussions in Ref. [1]; in particular, note that the American put option is always more expensive than the European put option, as indeed it must be since it has more choice; furthermore, the American put option, for small values of the stock price S, has the same slope as the payoff function, hence smoothly joining it.

From Ref. [1], the inequalities obeyed by the price of American call and put options C and P, respectively, on a stock with stock price S, strike price K and maturing at future time T is given by the following:

$$S - K \leqslant C - P \leqslant S - e^{-rT}K, \tag{74}$$

where r is the spot interest rate. The put–call inequality for American option of a stock is seen in Fig. 26 to hold for the numerical option prices.

All the basic features of the algorithm for pricing an American put option for equity are simplified expressions that appear in the more complex algorithm needed to price American options for interest rate instruments.

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