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Price of coupon bond options in a quantum field theory of forward interest rates

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Abstract

European options on coupon bonds are studied in a quantum field theory model of forward interest rates. A approximation scheme for finding the option price is developed based on the fact that the volatility of the forward interest rate is a small quantity. The field theory for the forward interest rates is in effect Gaussian, and when the payoff function for the coupon bonds option is included it makes the field theory exponentially nonlinear. A Feynman perturbation expansion gives a result for the price of Libor swaption that agrees quite well with the market price.

Keywords: Coupon bond option; Forward interest rates; Quantum field theory

1. Introduction

Coupon bonds and interest swaps are primary financial instruments, and options on coupon bonds and swaps are amongst the most widely traded of financial instruments. The Bank of International Settlements (Switzerland) estimated that in 2001 the notional value of the swap market was approximately US$40 trillion dollars and that of the combined interest rate caps and swaptions market was about US$9 trillion dollars.

Pricing of coupon bond options and swaptions (which will be shown to be equivalent to the coupon bond option) is an important topic in the field of interest derivatives [1]. The current practise in the market is to price swaptions using the Black formula for the pricing of interest rate caps [2]. In this paper an alternative derivation of the coupon bond option and of the swaption is presented [3]. The main advantages of the quantum field theory approach are the following: (a) in the field theory approach a single volatility function can price a coupon bond option, whereas in the Black formula each caplet in a swaption has its own volatility function; and (b) the correlations of the forward interest rates are parsimoniously accounted for in the field theory model [4].

2. Quantum field theory model of interest rates

Forward interest rates $f(t,x)$ are the interest rates, fixed at time t, for an instantaneous loan at future times $x > t$. The price at time t of a Treasury Bond that matures at some future time $T > t$—denoted by $B(t,T)$—is

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defined in terms of the forward interest rates as $B(t,T) = \mathrm{e}^{-\int_t^T \mathrm{d}x f(t,x)}$. Consider a zero coupon bond denoted by $B(t_*, T_i)$ that is going to be issued at time future time $t* > t_0$ and maturing at time T_i; the forward price of the bond, in terms of forward interest rates is defined by $F_i = F(t_0, t_*, T_i) = \mathrm{e}^{-\int_{t_*}^{T_i} \mathrm{d}x f(t_0,x)}$.

Let $A(t,x)$ be a two-dimensional quantum field, which drives the time evolution of forward interest rates $f(t,x)$, and is defined by

$$\frac{\partial f(t,x)}{\partial t} = \alpha(t,x) + \sigma(t,x)A(t,x),$$

where $\alpha(t,x)$ is the drift of the forward interest rates (fixed by a choice of numeraire), and $\sigma(t,x)$ is the volatility (fixed from the market). Both the forward interest rates $f(t,x)$ and its derived velocity field $A(t,x)$ are considered to be *two-dimensional quantum fields*; for each t and each x $f(t,x)$ (and $A(t,x)$) is an independent (random) integration variable.

The quantum field theory of the forward interest rates is defined by the generating (partition) function given by [4]

$$Z[h] = E\left[\mathrm{e}^{\int_{t_0}^\infty \mathrm{d}t \int_0^\infty \mathrm{d}z\, h(t,z)A(t,z)}\right] \equiv \frac{1}{Z}\int DA\mathrm{e}^{S[A]+\int_{t_0}^\infty \mathrm{d}t \int_0^\infty \mathrm{d}z\, h(t,z)A(t,z)}$$
$$= \exp\left(\frac{1}{2}\int_{t_0}^\infty \mathrm{d}t \int_0^\infty \mathrm{d}z\, \mathrm{d}z' h(t,z)D(z,z';t)h(t,z')\right)$$

and which yields the following correlator for the $A(t,x)$ quantum field

$$\langle A(t,x)A(t',x')\rangle = E[A(t,x)A(t',x')] = \delta(t-t')D(x,x';t).$$

The stiff Lagrangian that describes the evolution of the forward interest rates, for market time z, is determined by parameters μ, λ and η, and is written as [5]

$$\mathscr{L} = -\frac{1}{2}\left(A^2 + \frac{1}{\mu^2}\left(\frac{\partial A}{\partial z}\right)^2 + \frac{1}{\lambda^4}\left(\frac{\partial^2 A}{\partial z^2}\right)^2\right); \quad z = (x-t)^\eta : \text{Market time.}$$

The stiff action and partition function is given by[1]

$$S[A] = \int_0^\infty \mathrm{d}t \int_t^\infty \mathrm{d}x\, \mathscr{L}; \quad Z = \int DA\, \mathrm{e}^{S[A]}.$$

Discretize time $t = n\varepsilon$ into a lattice of points with spacing $\varepsilon = 1$ day. Then

$$\delta f(t,x) \equiv f(t+\varepsilon,x) - f(t,x) = \varepsilon\alpha(t,x) + \varepsilon\sigma(t,x)A(t,x).$$

For $\theta = x - t, \theta' = x' - t$ the equal time correlator is given by

$$E(A(t,\theta)A(t,\theta')) \equiv \langle A(t,\theta)A(t,\theta')\rangle = \frac{1}{\varepsilon}D(\theta,\theta')$$

$$\Rightarrow \langle \delta f(t,\theta)\delta f(t,\theta')\rangle_c = \varepsilon\sigma(\theta)D(\theta,\theta')\sigma(\theta').$$

For market time $z(\theta) = \theta^\eta$ the market correlator of the model propagator $D(z(\theta), z(\theta'))$ yields the following:

$$\langle \delta f(t,\theta)\delta f(t,\theta')\rangle_c = \varepsilon\sigma(z(\theta))D(z(\theta),z(\theta'))\sigma(z(\theta')).$$

The Libor market value of $\langle \delta f(t,\theta)\delta f(t,\theta')\rangle_c$ and its best fit by the model, namely by $\sigma(z(\theta))D(z(\theta),z(\theta'))\sigma(z(\theta'))$, is given in Fig. 1. The best fit of the stiff propagator's parameters is given in Fig. 2 and yields [5] $(\lambda z = [\tilde{\lambda}\theta]^\eta, \mu z = [\tilde{\mu}\theta]^\eta)$ $\tilde{\lambda} = 1.79/\text{year}; \tilde{\mu} = 0.90/\text{year}; \eta = 0.34$. Root mean square error for the entire fit: 0.4%.

Note that although one is formally using a Gaussian action for the 'stiff' model of the forward interest rates, by introducing market time $z(\theta) = \theta^\eta$ one in effect has rendered the theory nonlinear. Put another way, if one

[1] For the rest of the paper for simplicity of notation $\eta = 1$ and $z = x - t$.

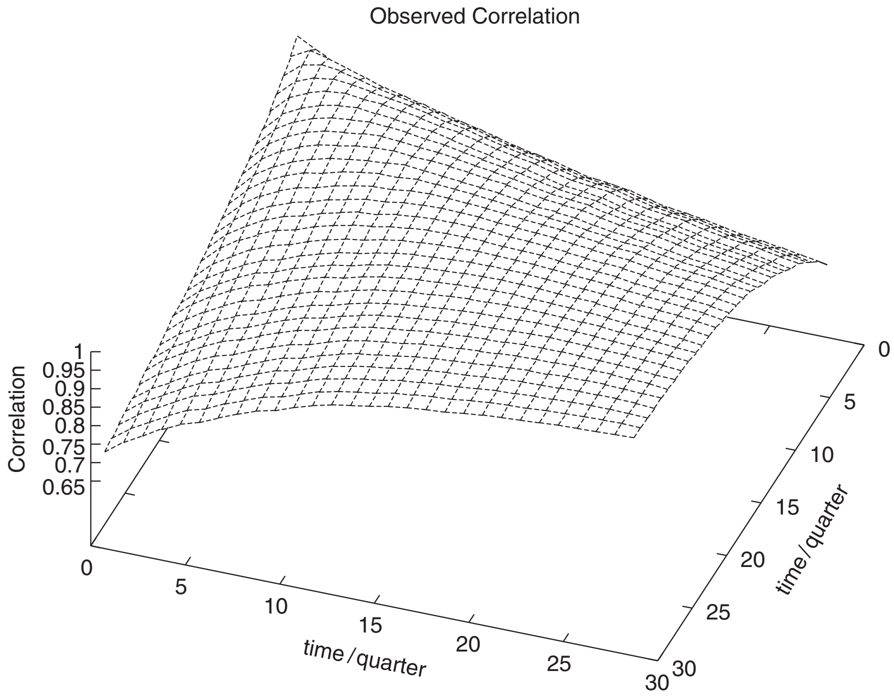

Fig. 1. The Libor market value of the normalized correlator $D(\theta, \theta')$.

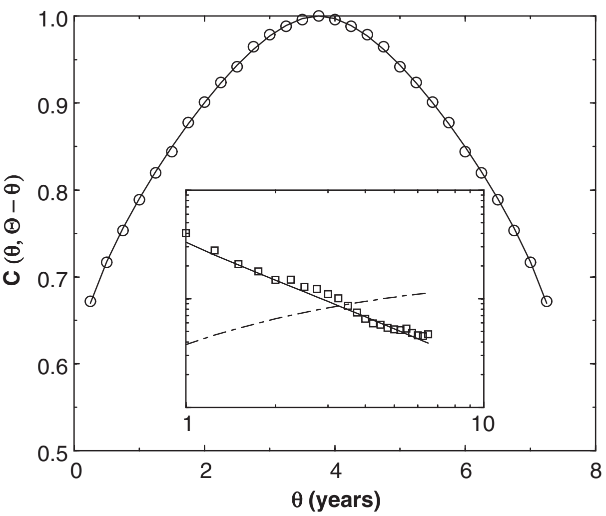

Fig. 2. Figure showing the fitted 'stiff' propagator for the value of the propagator $D(z(\theta), z(\theta'))$ with market time $z(\theta) = \theta^{\eta}$ and along the diagonal $\theta = -\theta'$. The inset shows the curvature orthogonal to the diagonal, with the dashed line showing the curvature for $\eta = 0$.

uses calendar time $\theta = x - t$, then one would need a non-Gaussian stiff action to obtain the propagator given by $D(z(\theta), z(\theta'))$. The 'fat tail' problem does not appear in the forward interest rates; what does appear, however, is that the forward rates have nontrivial skewness and kurtosis that cannot be explained by a Gaussian model.

3. Coupon bond option

Consider a coupon bond on a principal L which matures at time T and that pays fixed dividends (coupons) a_i at times $T_i, i = 1, 2, \ldots, N$; the time of maturity of the coupon bond is given by $T_N = T$. The value of the coupon bond at time $t_* < T_i$ is given by [6]

$$\sum_{i=1}^{N} a_i B(t_*, T_i) + LB(t_*, T) = \sum_{i=1}^{N} c_i B(t_*, T_i),$$

where the final payment is included in the sum by setting $c_i = a_i,\ c_N = a_N + L$.

Consider a European call option that matures at time t_* and has a strike price K. The price of the option, at time $t_0 < t_*$, is given by discounting the value of the payoff function $(\sum_{i=1}^{N} c_i B(t_*, T_i) - K)_+$ [6] using the spot interest rate $r(t) = f(t, t)$; hence

$$C(t_0, t_*, K) = E_{\text{Money Market}}\left[\mathrm{e}^{-\int_{t_0}^{t_*} \mathrm{d}t\, r(t)} \left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+ \right]$$

The money market drift is given by

$$\alpha(t, x) = \int_t^x \mathrm{d}x' M(x, x'; t);\ \ M(x, x'; t) \equiv \sigma(t, x) D(x, x'; t) \sigma(t, x').$$

4. Coupon bond option price

The fundamental idea in evaluating the price of the coupon bond option is to perturb the price about the forward coupon bond price [3]. The price of the coupon bond can be re-written as

$$\sum_{i=1}^{N} c_i B(t_*, T_i) = \sum_{i=1}^{N} c_i F_i + \sum_{i=1}^{N} c_i [B(t_*, T_i) - F_i] \equiv F + V; \quad V = \sum_{i=1}^{N} c_i [B(t_*, T_i) - F_i].$$

Note the forward price of the bond portfolio is F and the perturbation term V, which will be shown to be small perturbation about F.

The bond payoff function can be re-written using the properties of the Dirac delta function. Since $\delta(W) = (1/2\pi) \int_{-\infty}^{+\infty} \mathrm{d}\eta\, \mathrm{e}^{\mathrm{i}\eta W}$ it follows that

$$\left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+ \equiv (F + V - K)_+ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}W\, \mathrm{d}\eta\ \mathrm{e}^{\mathrm{i}\eta(V - W)} (F + W - K)_+.$$

Hence the price of the call option can be re-written as

$$C(t_0, t_*, K) = B(t_0, t_*) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}W\, \mathrm{d}\eta (F + W - K)_+ \mathrm{e}^{-\mathrm{i}\eta W} Z(\eta),$$

where the partition function, using the forward martingale measure [6], is given by

$$Z(\eta) = \langle \mathrm{e}^{\mathrm{i}\eta V} \rangle_F = \frac{1}{Z} \int DA \mathrm{e}^{\mathrm{i}\eta V} \mathrm{e}^{S[A]}; \quad Z = \int DA \mathrm{e}^{S[A]}.$$

From the expression for the partition function, the effective action for the pricing of the coupon bond option is given by

$$\begin{aligned} S_{\text{Eff}} &\equiv S[A] + \mathrm{i}\eta V \\ &= S[A] + \mathrm{i}\eta \sum_{i=1}^{N} J_i [\mathrm{e}^{-\int_{R_i} \alpha} \mathrm{e}^{-\int_{R_i} \sigma A} - 1]. \end{aligned}$$

The action S_{Eff} is nonlocal and has exponential nonlinearity. A Feynman perturbation expansion has been obtained for the price of a swaption in Ref. [3], and the result of the calculation is reported in the next section.

4.1. Swaps and swaptions

A swap of the first kind is one in which a party pays at fixed rate R_S and receives payments at the floating rate. Hence, at time T_n the value of the swap is the difference between the floating payment received at the rate of Libor $L(t, T_n)$, and the fixed payments paid out at the rate of R_S. Similarly, a swap of the second kind is one in which the party holding the swap pays at the floating rate and receives payments at fixed rate R_S.

Consider a swap that starts at time T_0 and ends at time $T_N = T_0 + N\ell$, with payments both fixed and floating payments being made at times $T_0 + n\ell$, with $n = 1, 2, \ldots, N$, where $\ell = 90$ days. The payoff function of the swaption is given by [6,7]

$$S(T_0, R_S) = V\left[1 - B(T_0, T_0 + N\ell) - \ell R_S \sum_{n=1}^{N} B(T_0, T_0 + n\ell)\right]_+ .$$

One can see that a swaption is equivalent to an option on a portfolio of coupon bonds.

A perturbation expansion for the price of a swaption is possible because for the Libor market $\sigma(t, x) \simeq 10^{-2}\,\text{year}^{-3/2}$ [4]. There are *no divergences* in the Feynman diagrams for the swaption price, since, unlike relativistic quantum field theories, the propagator has no short distance singularities at $x = x'$ because $M(x, x; t) = \sigma^2(t, x)$.

The traded price of a swaption on Libor and the model price calculated from the quantum field theory model is compared in Fig. 3, and shows that the market price and field theory model price for a 2×10 swaption on the UD$ agree quite well.

5. Conclusions

The stiff quantum field theory model for forward interest rates describes historical Libor data to an accuracy of over 99%. Coupon bond options are described quite well by a highly nonlinear, but tractable, two-dimensional quantum field theory. A Feynman perturbation expansion of the coupon bond option price is possible because the volatility of the forward interest rates $\sigma^2(t, x)$ is a small parameter.

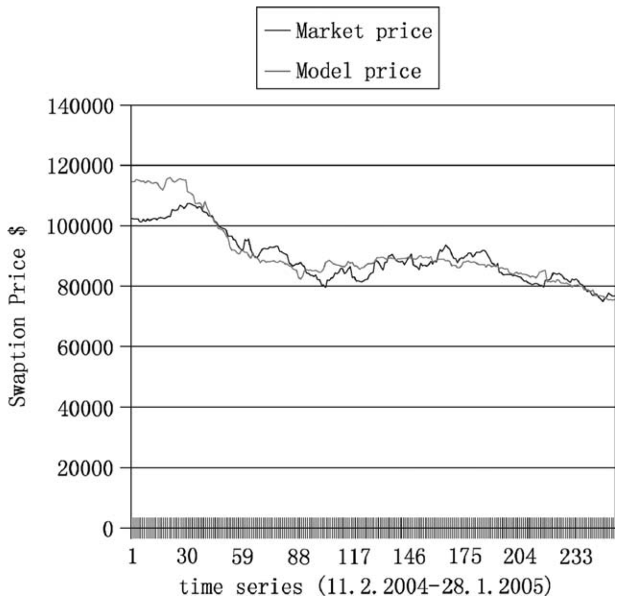

Fig. 3. The price, from the Libor market and from the model, of a US$ 2×10 swaption that matures in 2 years and consists of payments running for 10 years for the period 2003–2005.

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