Question: 1/5

Write answers on this side of the paper only.

(a) Since \((\mathbf{\Phi}^T, \mathbf{A})\) are 4-vector fields, their Lorentz scalar product,

\[ (-\mathbf{c}, \mathbf{F}^T)(\mathbf{\Phi}, \mathbf{A}) = -c \left[ \mathbf{p} \mathbf{\Phi} - \frac{1}{c} \mathbf{j} \cdot \mathbf{A} \right], \]

is a 4-scalar field.

(b) The gauge transformation \(\mathbf{A} \rightarrow \mathbf{A} - \mathbf{\nabla} \lambda, \mathbf{\Phi} \rightarrow \mathbf{\Phi} + \frac{1}{c} \mathbf{E} \mathbf{k} \)

adds

\[ \int (c^2 \frac{d}{dt} \lambda + \frac{1}{c} \mathbf{j} \cdot \mathbf{\nabla} \lambda) \]

\[ = \int \left( \int \left( \frac{d}{dt} \left( \frac{1}{c} \mathbf{j} \cdot \mathbf{\nabla} \lambda \right) + \int \left( \frac{d}{dt} \left( \frac{1}{c} \mathbf{j} \cdot \mathbf{\nabla} \lambda \right) \right) \right)_\mathbf{a} \mathbf{l} \]

\[ = \int \int \left( \frac{d}{dt} \left( \frac{1}{c} \mathbf{j} \cdot \mathbf{\nabla} \lambda \right) \right) \lambda \mathbf{p}, \]

which is a total time derivative, indeed.

(2) (a) With the mirror moving in the \(z\) direction, and the wave vectors of the incoming and reflected light in the \(x, y\) plane, we have

\[ \left( \frac{\mathbf{v}^c}{c} \right) \] for the 4-velocity of the mirror,

and \( \omega \mathbf{c} \left( \sin \theta \mathbf{c}, \frac{\omega}{c} \mathbf{c} \right) \) for the
4-wave vectors of the plane waves. In the rest-frame of the mirror we would have
\[
\left( \begin{array}{c} 0 \\ \frac{1}{c} \end{array} \right)
\]
for the 4-velocity, and
\[
\frac{\omega_0}{c} \left( \begin{array}{c} \sin \theta_0 \\ \cos \theta_0 \end{array} \right)
\]
for the two 4-wave vectors.

The Lorentz transformation into the rest-frame of the mirror is
\[
\left( \begin{array}{c} ct \\ x \\ y \\ z \end{array} \right) \rightarrow \left( \begin{array}{cccc} 1 & 0 & 0 & -\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v}{c} & 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} ct \\ x \\ y \\ z \end{array} \right),
\]

as verified by
\[
\left( \begin{array}{c} 0 \\ 0 \\ \frac{v}{c} \end{array} \right) \rightarrow c \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ \frac{v}{c} \end{array} \right).
\]

So,
\[
\frac{\omega_0}{c} \left( \begin{array}{c} \sin \theta_0 \\ \cos \theta_0 \end{array} \right) = \gamma \left( \begin{array}{cccc} 1 & 0 & 0 & -\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v}{c} & 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} \omega_0 \\ 0 \\ 0 \\ \frac{v}{c} \end{array} \right)
\]

\[
= \gamma \left( \begin{array}{c} \frac{\omega_0}{c} \\ 0 \\ 0 \\ -\frac{v}{c} \end{array} \right),
\]

and, likewise,
\[
\frac{\omega_0}{c} \left( \begin{array}{c} \sin \theta_0 \\ \cos \theta_0 \end{array} \right) = \gamma' \left( \begin{array}{cccc} 1 & 0 & 0 & \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v}{c} & 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} \omega_0 \\ 0 \\ 0 \\ \frac{v}{c} \end{array} \right),
\]

and upon equating the time components and
the z components, we have

\[ \omega (1 + \frac{\nu}{c} \cos \nu) = \omega' (1 - \frac{\nu}{c} \cos \nu'), \]
\[ \omega (\cos \nu + \frac{\nu}{c}) = \omega' (\cos \nu' - \frac{\nu}{c}). \]

These imply

\[ \omega' = \omega \sqrt{1 + \frac{2\nu}{c^2} \cos \nu + \left( \frac{\nu}{c} \right)^2} \]
and \( \cos \nu' = \frac{\cos \nu + \nu}{1 + \nu \cos \nu} \) with \( \nu = \frac{2\nu}{c} \sqrt{1 + (\frac{\nu}{c})^2} \).

(b) When \( \frac{\nu}{c} \to 1 \), then \( \omega' \to \infty \) and \( \cos \nu' \to 1 \), or \( \nu' \to 0 \).

(c) When \( \nu = -c \cos \theta \), then the mirror recedes at the speed by which the incoming light approaches the mirror, and there should then be no reflection, so that \( \omega = \omega' \) and \( \frac{\nu'}{\nu} = 1 - \frac{\nu}{c} \) is expected. This is also what the equations tell:

\[ \omega' = \omega \frac{1}{1 - \frac{2c}{c^2} \cos \theta + (\frac{\nu}{c})^2} = \omega, \]
\[ \cos \nu' = \frac{\cos \theta - \frac{2c \cos \theta}{1 + (c \cos \theta)^2}}{1 - \frac{2c \cos \theta}{1 + (c \cos \theta)^2} = - \cos \theta = \cos (\pi - \theta)}, \]

indeed.
(a) For \( \psi = \frac{\pi}{2} \), we have \( \cos \psi = 0 \), and this requires \( u = -\cos \theta \), or

\[
1 + 2 \frac{\nu/c}{\cos \psi} + \left( \frac{\nu}{c} \right)^2 = 0,
\]

solved by

\[
\frac{\nu}{c} = \frac{-1}{\cos \psi} \pm \sqrt{\frac{1}{\cos^2 \psi} - 1}
\]

\[
= -\frac{1 \pm \sin \theta}{\cos \psi} = -\frac{1 \pm \sin \theta}{1 \mp \sin \theta}
\]

\[
= -\sqrt{\frac{1 \pm \sin \theta}{1 \mp \sin \theta}}
\]

where only the upper sign is physically possible, so that

\[
\nu = -\sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} \quad c = -\frac{1 - \sin \theta}{\cos \psi} c
\]

is the answer.

(b) Since \( \nabla \cdot \mathbf{J}(r, t) = -\frac{\partial}{\partial r} \mathbf{P}(r, t) = \mathbf{A}(t) \cdot \nabla \delta(r) \),

we have \( \mathbf{J}(r, t) = \mathbf{A}(t) \delta(r) + \text{[a curl, possibly]} \),

but as part (b) and (c) show, there is actually no need to add a curl of something.

(b) Without the extra curl:

\[
\mu(t) = \frac{1}{2c} \int (\mathbf{A}(t) \cdot \mathbf{A}(t)) \delta(r) \, dr = 0.
\]

(c) Without the extra curl,
\[
\int d^2 \mathbf{r} \; \mathbf{f}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{d}(t), \quad \text{as it should be}
\]
and the extra cubic term would not matter.

(d) It is
\[
\mathbf{A}(\mathbf{r}, t) = \int d^2 \mathbf{r}' \int dt' \; \frac{\delta(t-\frac{t'}{c} - \frac{\mathbf{r} - \mathbf{r}'}{c}) \mathbf{\hat{z}}(t')}{\mathbf{r}' - \mathbf{r}'} + \frac{1}{c} \mathbf{\dot{d}}(t) \delta(\mathbf{r})
\]
\[
= \frac{1}{\tau c} \mathbf{\dot{d}}(t - \tau/c).
\]

4. Larmor says:
\[
\frac{d^2}{dt^2} \mathbf{d}(t) = \frac{1}{4\pi c^3} \left[ \mathbf{\hat{n}} \times \mathbf{\ddot{d}}(t) \right]^2
\]
where \(\mathbf{\ddot{d}}(t) = \mathbf{\hat{n}} \times (\mathbf{\omega} \times \mathbf{\dot{d}}(t)) = -\omega^2 \mathbf{\dot{d}}(t),\)

and \(\left| \mathbf{\hat{n}} \times \mathbf{\dot{d}}(t) \right|^2 = \mathbf{\dot{d}}(t)^2 - (\mathbf{\hat{n}} \cdot \mathbf{\dot{d}}(t))^2\)

\[
\text{averaged over one period} \quad d^2 - \frac{1}{2} (\sin \theta)^2 d^2,
\]

so that
\[
\frac{dP}{d\Omega} = \frac{\omega^4 d^2}{4\pi c^3} \left[ 1 - \frac{1}{2} (\sin \theta)^2 \right],
\]

and the total power is
\[
P = \frac{\omega^4 d^2}{c^3} \left( 1 - \frac{1}{2} \frac{2}{3} \right) = \frac{2}{3} \frac{\omega^4 d^2}{c^3},
\]

which we get by integration over the solid angle of \(4\pi\), or alternatively from
\[
\frac{dP}{d\sigma} = \frac{2}{3 c^3} \left| \mathbf{\dot{d}} \right|^2.
\]