1. Mean velocity (10 marks)
Three-dimensional motion: position vector operator \( \vec{R} \), momentum vector operator \( \vec{P} \). The system is in an eigenstate of the Hamilton operator \( H(\vec{P}, \vec{R}) \). Show that the mean velocity, that is: the expectation value of the velocity vector operator \( \vec{V} = \frac{d}{dt} \vec{R} \), is zero.

2. Time-dependent spreads (25=12+8+5 marks)
At time \( t = 0 \), the initial position wave function of a one-dimensional harmonic oscillator (position operator \( X \), momentum operator \( P \), mass \( M \), circular frequency \( \omega \)) is given by
\[
\psi(x) = \sqrt{\kappa} e^{-\kappa |x|}
\]
with \( \kappa > 0 \).
(a) Determine \( \delta X(t) \) and \( \delta P(t) \), the time-dependent spreads in position and momentum, respectively.
(b) Verify that Heisenberg’s position-momentum uncertainty relation is obeyed at all times.
(c) For which value of \( \kappa \) is the uncertainty product \( \delta X(t) \delta P(t) \) independent of time \( t \)?

3. Orbital angular momentum (20=5+10+5 marks)
The Hamilton operator of a spinning top is
\[
H = \frac{1}{2I_1} L_1^2 + \frac{1}{2I_2} L_2^2 + \frac{1}{2I_3} L_3^2,
\]
where \( L_1, L_2, L_3 \) are the cartesian components of the angular momentum vector operator \( \vec{L} \), and \( I_1, I_2, I_3 \) are the moments of inertia for the three major axes of rotation.
(a) State the equation of motion obeyed by \( L_1(t) \).
(b) If the top is in a common eigenstate of \( \vec{L}_2 \) and \( L_3 \) with eigenvalues \( 2\hbar^2 \) and \( \hbar \), respectively, what is the expectation value \( \langle H \rangle \) of \( H \) and what is its spread \( \delta H \)?
(c) If \( I_2 = I_3 \), what are the eigenvalues of \( H \)?
4. Hydrogen-like atoms (20=8+8+4 marks)
You have a tritium atom (\(^3\)H, nuclear charge \(Z = 1\)) in its ground state [principal quantum number \(n = 1\), angular momentum quantum numbers \((l, m) = (0, 0)\)]. Suddenly the triton nucleus undergoes a \(\beta\) decay whereby the emitted electron (and also the neutrino) escape so rapidly that we can regard the net effect as an instantaneous replacement of the triton by a \(^3\)He nucleus (nuclear charge \(Z = 2\)). For the bound electron, this amounts to a sudden doubling of the nuclear charge.

(a) What is the probability that, after the decay, the resulting \(^3\)He\(^+\) ion is found in its electronic ground state as well?

(b) What is the probability that you find the \(^3\)He\(^+\) ion in its excited state with \(n = 2\) and \(l = 0\)?

(c) What is the probability that you find the \(^3\)He\(^+\) ion in one of its exited states with \(n = 2\) and \(l = 1\)?

Hint: For hydrogenic wave functions see equations (5.2.27), (6.7.6), and (6.7.16) in the lecture notes.

5. Perturbation Theory (25=15+10 marks)
A harmonic oscillator (ladder operators \(A, A^\dagger\); Hamilton operator \(H_0 = \hbar \omega A^\dagger A\)) is perturbed by \(H_1 = \hbar \Omega [A^\dagger (AA)^{-1/2} + (AA)^{-1/2} A]\). We denote the \(n\)th eigenvalue of the total Hamilton operator \(H = H_0 + H_1\) by \(E_n = \hbar \omega \epsilon_n(\Omega/\omega)\) where, of course, the unperturbed energies \(E^{(0)}_n = n \hbar \omega\) are recovered by \(\epsilon_n(0) = n\) for \(n = 0, 1, 2, \ldots\)

(a) For \(n = 0, 1, 2, \ldots\), determine \(\epsilon_n(\Omega/\omega)\) up to 2nd order in \(\Omega/\omega\) by Rayleigh–Schrödinger perturbation theory.

(b) Find the 2nd-order approximation to \(\epsilon_0(\Omega/\omega)\) in Brillouin–Wigner perturbation theory (only \(n = 0\) here).