Problem 1 (8+12+10=30 marks)
Consider the one-dimensional motion (position operator $X$, momentum operator $P$) that is governed by the Hamilton operator
\[ H = vP + \frac{1}{2}\kappa^2 X^2 \]
where $v$ and $\kappa$ are positive constants.

(a) State and solve the equations of motion obeyed by $X(t)$ and $P(t)$.

(b) Determine the time transformation function $\langle x, t | p, t_0 \rangle$.

(c) Given the expectation values $\langle X \rangle = 0$, $\langle P \rangle = 0$, $\langle XP \rangle = \frac{1}{2}i\hbar$, $\langle X^2 \rangle = x_0^2$, and $\langle P^2 \rangle = p_0^2$ at time $t_0$, find the spreads $\delta X(t)$ and $\delta P(t)$ at time $t$.

Problem 2 (10 marks)
Harmonic oscillator ladder operators $A, A^\dagger$; coherent state kets $|a\rangle$ and bras $\langle a^*|$.
Show that $e^{zA^\dagger}|a\rangle = |a+z\rangle$ for all complex numbers $z$. Which bra results from $\langle a^*|e^{zA}?$

Problem 3 (10 marks)
A three-dimensional system (mass $M$, position vector operator $\vec{R}$, momentum vector operator $\vec{P}$) is in an eigenstate of the Hamilton operator $H = \vec{P}^2/(2M) + V(\vec{R})$. Show that the mean torque, that is: the expectation value of the torque vector operator $\vec{R} \times \vec{F}$, is zero, where $\vec{F} = -\frac{\partial}{\partial \vec{R}}V(\vec{R})$ is the force vector operator.

Problem 4 (5+20=25 marks)
Orbital angular momentum vector operator $\vec{L}$ with cartesian components $L_1, L_2$, and $L_3$; as usual, $|l, m\rangle$ is a joint eigenket of $L^2$ and $L_3$.

(a) For $l = 1$ and a given value for the angle parameter $\gamma$, what are the eigenvalues of $L_1 \cos \gamma + L_2 \sin \gamma$?

(b) Find the respective eigenkets of $L_1 \cos \gamma + L_2 \sin \gamma$ as superpositions of the eigenkets of $L_3$.

Problem 5 (25 marks)
One simplifying assumption in the derivation of the Bohr energies of hydrogen-like atoms is that we regard the atomic nucleus as a point charge. When we want to describe it rather as a homogeneously charged ball, we replace the potential energy $V_{\text{point}}(r) = -\frac{Ze^2}{r}$ by $V_{\text{sphere}}(r)$:
\[ V_{\text{sphere}}(r) = \begin{cases} \frac{Ze^2}{2b^3}(3b^2 - r^2) & \text{for } r < b, \\ -\frac{Ze^2}{r} & \text{for } r > b, \end{cases} \]
where $b > 0$ is the radius of the ball-shaped nucleus. Bear in mind that $b$ is a tiny fraction of the Bohr radius $a_0$ (roughly $b/a_0 \approx 10^{-4}$), and determine the resulting 1st-order change in the atomic ground-state energy. — Hint: For hydrogenic wave functions see equations (5.2.27), (6.7.6), and (6.7.16) in the lecture notes.