Problem 1 (15=9+6 points)
Operator $A$ has three different eigenvalues: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$.

(a) Write the projection operators $|a_j⟩⟨a_j| = δ(A, a_j)$ ($j = 1, 2, 3$) as polynomials in $A$ of the lowest possible degree.

(b) Write the operator function $f(A) = \log_2 A$ as a polynomial in $A$ of the lowest possible degree.

Problem 2 (35=15+5+10+5 points)
Operators $U$ and $V$ are two cyclic unitary operators of period $N$ for an $N$-dimensional quantum degree of freedom.

(a) We know from the lecture and the tutorials that, for all integer values of $n$ and $m$,

$$\text{tr}\{U^m V^n\} = \begin{cases} N & \text{if } U^m = 1 \text{ and } V^n = 1, \\ 0 & \text{otherwise}, \end{cases} \quad (*)$$

if $U$ and $V$ are a pair of complementary observables. Now show the converse: If (*) holds, then $U$ and $V$ are a complementary pair. — Hint: Recall how to express a squared bracket in terms of a trace.

(b) Now consider the set of $N + 1$ operators defined by

$$W_0 = U, \ W_1 = U V, \ W_2 = U^2 V, \ldots, \ W_{N-1} = U^{N-1} V, \ W_N = V,$$

where $U$ and $V$ are the usual pair of complementary unitary operators. For $j = 1, 2, \ldots, N - 1$, show that $W_j = U^j V$ is unitary and $W_j^N$ is a multiple of the identity.

(c) Then show that each pair $W_j, W_k$ ($0 \leq j < k \leq N$) is a complementary pair if $N$ is prime.

(d) For $N = 4$, find a pair $W_j, W_k$ that is not a complementary pair.
**Problem 3** (20=10+10 points)

Mass $M$ moves along the $x$ axis whereby the Hamilton operator

$$H = \frac{1}{2M}p^2 - \lambda \delta(X - x_0)$$

with constant $\lambda$ and $x_0$ governs the evolution. The time transformation function $\langle x, t_1 | x', t_2 \rangle$ depends on the strength $\lambda$ of the coupling to, and the location $x_0$ of, the delta-function potential $\delta(X(t) - x_0) = |x_0, t\rangle \langle x_0, t|.$

(a) Use the quantum action principle to express $\frac{\partial}{\partial \lambda} \langle x, t_1 | x', t_2 \rangle$ as an integral over the intermediate time $t$.

(b) Then recall the $\lambda = 0$ form of the time transformation function and determine the value of $\frac{\partial}{\partial \lambda} \langle x_0, t_1 | x_0, t_2 \rangle \big|_{\lambda = 0}$. — Hint: A parameterization that we used in lecture for an integral on page 40 of the notes could be useful.

**Problem 4** (30=10+10+10 points)

For an operator $A$ that has an inverse $A^{-1}$, one can define the determinant by the differential statement

$$\delta \det \{A\} = \det \{A\} \text{tr} \{A^{-1} \delta A\} \quad (***)$$

together with $\det \{1\} = 1$.

(a) Consistency requires that $\delta_1 \delta_2 \det \{A\} = \delta_2 \delta_1 \det \{A\}$ if $\delta_1$ and $\delta_2$ symbolize two independent infinitesimal variations. Verify that this is indeed correct. — Hint: You will need an expression for $\delta A^{-1}$.

(b) Use (***) to show that $\det \{AB\} = \det \{A\} \det \{B\}$.

(c) Use (***), to express $\det \{e^Z\}$ in terms of $\text{tr} \{Z\}$.  