The \( y(x) \) that minimizes the integral over \( y(x)^2 \) obeys the differential equation

\[
y''(x) = -6x(1-x^2),
\]

where \( \lambda \) is the Lagrange multiplier for the constraint and the factor of \(-6\) is for convenience. The solution that takes \( y(\pm a) = 0 \) into account is

\[
y(x) = \lambda (a^3 - 1x^2),
\]

with the value of \( \lambda \) determined by

\[
a^3 = \int_{-a}^{a} dx \, x(1-x^2)
\]

\[
= \frac{3}{5} a^5,
\]

so that \( \lambda = \frac{5}{3a^2} \).

The minimal value is, therefore,

\[
\int_{-a}^{a} dx \, (-3 \lambda x(1-x^2)) = 18\lambda^2 \int_{0}^{a} dx \, x^4
\]

\[
= \frac{18}{5} \lambda^2 a^5 = \frac{18}{5} \left( \frac{5}{3a^2} \right)^2 a^5 = 10 a.
\]
2 (a) Closure is obvious (if not: see below, page 4).
Neutral element: \( e = (1, 0) \).
Inverse element: \( g^{-1} = (a^*, -b) \).

Association: For \( g_1(g_2g_3) = (g_1g_2)g_3 \) we need
\[
a_1 (a_2 a_3 + b_2^* b_3) + b_1^* (b_2 a_3 + a_2^* b_3)
= (a_1 a_2 + b_1^* b_2) a_3 + (b_1 b_2 + a_1^* b_2) b_3
\]

and
\[
(b_1 (a_2 a_3 + b_2^* b_3) + a_1^* (b_2 a_3 + a_2^* b_3)
= (b_1 a_2 + a_1^* b_2) a_3 + (a_1 b_2 + b_1^* b_2) b_3;
\]
both are identities indeed.

(b) For \( a = a_1 a_2 + b_1^* b_2 \), \( b = b_1 a_2 + a_1^* b_2 \) we have
(i, ii) \( \text{Im} (a + b) = \text{Im} ((a_1 + b_1) a_2 + (a_1^* + b_2^*)) \)
\[
= \text{Im} (a_1 + b_1) \text{Re} (a_2 + b_2) + \text{Re} (a_1 + b_1) \text{Im} (a_2 + b_2).
\]
Therefore, if (i) \( \text{Im} a_1 = \text{Im} b_1 \) and \( \text{Im} a_2 = \text{Im} b_2 \) then also \( \text{Im} a = \text{Im} b \);
and if (ii) \( \text{Im} a_1 = -\text{Im} b_1 \) and \( \text{Im} a_2 = -\text{Im} b_2 \) then also \( \text{Im} a = -\text{Im} b \).
Accordingly there is closure under (i) and (ii). Further, $\text{Im} a = \text{Im} b = 0$ for the neutral element, so that $\text{Im} a = \pm \text{Im} b$ for $e$; and, finally, if $\text{Im} a = \pm \text{Im} b$ for $g = (a, b)$, then also for $g^{-1} = (a^*, -b)$.

Conclusion: (i) and (ii) define subgroups.

(iii) For $g_1 = (-i\sqrt{2}, 1), g_2 = (i\sqrt{2}, 1)$, we have $g_1 g_2 = (3, i\sqrt{8})$, so that $\text{Im} b \neq 0$ although $\text{Im} b = \text{Im} b_2 = 0$. This example therefore demonstrates the lack of closure. Conclusion: (iii) does not define a subgroup.

(iv) $b_1 = b_2 = 0$ implies $b = 0$, so that restriction (iv) defines a subgroup.

(c) The subgroup for (iv) is abelian, those for (i) and (ii) are not.

Case for (iv): $G = (9, 1, 0), G_2 = (9, 0, 0)$ give $G_1 G_2 = (9, 9, 0) = G_2 G_1$.

Case for (i) take $G_1 = (1+i, i), G_2 = (1+i, 1)$ to show that $G_1 G_2 \neq G_2 G_1$; and similarly for (ii).
Question 4.6

Write answers on this side of the paper only.

Returning to (a) closure: We would need to verify that

\[ |a_1, a_2 + b_1, b_2| = |b_1, a_2, a_1 + b_2| = 1 \]

If \(|a_1|^2 - |b_1|^2 = 1\) and \(|a_2|^2 - |b_2|^2 = 1\).

See:

\[
\begin{align*}
1 & a_{11}^2 |a_2|^2 + 1 b_{12}^2 |b_2|^2 - 1 b_{11}^2 |a_2|^2 - 1 a_{12}^2 |b_2|^2 \\
+ & a_{21}^2 |a_1|^2 + a_{12}^2 |b_1|^2 - b_{21}^2 |a_1|^2 - b_{12}^2 |b_1|^2 \\
\hline
\end{align*}
\]

\[ = (|a_1|^2 - |b_1|^2) (|a_2|^2 - |b_2|^2) = 1, \text{ indeed.} \]

We have \( f(t + T/2) = -f(t) \) and \( f(t) = 1 \) for \( 0 < t < T/2 \), so that

\[ F(s) = \int_0^{T/2} dt e^{-st} f(t) = \sum_{k=0}^{\infty} \int_0^{T/2} dt e^{-st} f(t + kT/2) \]

\[ = \sum_{k=0}^{\infty} \int_0^{T/2} dt e^{-s(t + kT/2)} f(t + kT/2) \]

\[ = \sum_{k=0}^{\infty} (-1)^k e^{-k \alpha T/2} \int_0^{T/2} dt e^{-st} \]

\[ = \frac{1}{1 + e^{-\alpha T/2}} \]

\[ = \frac{1 - e^{-\alpha T/2}}{2} \]

\[ = \frac{1}{2} \tanh (\alpha T/4). \]
Question 5/6

Write answers on this side of the paper only.

\[ 4 \] (a) We have

\[
\int \frac{dz}{2mz} \ z^n \ e^{\frac{1}{2}t \left( z - \frac{1}{z} \right)}
\]

\[
= \left\{ \begin{array}{ll}
\int \frac{dz}{2mz} \ \sum_{m=-\infty}^{\infty} z^{m+n-1} b_m(t) = b_{-n}(t) \\
\int \frac{dz}{2mz} \ \sum_{m=-\infty}^{\infty} (-1)^m b_m(t) = (-1)^n b_n(t)
\end{array} \right.
\]

Since \( \int \frac{dz}{2mz} \ z^{-k-1} = \left\{ \begin{array}{ll} 1 & \text{if } k = 0 \\
0 & \text{if } k = \pm 1, \pm 2, \pm 3, \ldots
\end{array} \right. \)

(b) We take \( n = 0, 1, 2, \ldots \) and get

\[ (-1)^n B_n(t) = B_{-n}(t) \]

\[
= \int dt \ e^{-st} \int \frac{dz}{2mz} \ z^n \ e^{\frac{1}{2}t \left( z - \frac{1}{z} \right)}
\]

\[
= \int \frac{dz}{2mz} \ z^n \ \frac{1}{\lambda - \frac{1}{2} \left( z - \frac{1}{z} \right)}
\]

\[
= \int \frac{dz}{2mz} \ \frac{2 z^n}{1 + 2 z^2 - z^2}
\]

\[
= \int \frac{dz}{2mz} \ \frac{2 z^n}{(z^3 - z^2)(z^2 - z)}
\]
Question 6/6

Write answers on this side of the paper only.

with \[ z_1 + z_2 = -1 \] and \[ z_1 + z_2 = 2 \]

or \[ z_1 = \alpha + \sqrt{1 + \alpha^2} \]

and \[ z_2 = \alpha - \sqrt{1 + \alpha^2} \]

Since \( z_1 > 1 \) and \( -1 < z_2 < 0 \) for \( \alpha > 0 \), the pole at \( z_1 \) is outside the unit circle whereas the pole at \( z_2 \) is inside. In terms of the residue at \( z = z_2 \), we thus get

\[
\left(-1\right)^n B_n(0) = B_{-n}(0) = \frac{2 \pi n}{z_1 - z_2} = \frac{(\alpha - \sqrt{1 + \alpha^2})^n}{\sqrt{1 + \alpha^2}}
\]

so that

\[
B_n(0) = (-1)^n B_{-n}(0) = \frac{(\sqrt{1 + \alpha^2} - \alpha)^n}{\sqrt{1 + \alpha^2}}
\]

for \( n = 0, 1, 2, \ldots \).

(C) We have

\[
\int_0^\infty dt \, k_0(t) = B_0(0) = 1
\]

and

\[
\int_0^\infty dt \frac{B(t)}{t} = \int_0^\infty ds \, B_1(s)
\]

\[
= \int_0^\infty ds \left( 1 - \frac{\alpha}{\sqrt{1 + \alpha^2}} \right) = \left( \alpha - \sqrt{1 + \alpha^2} \right) \bigg|_{\alpha = 0}^\infty = 1
\]