Problem 1 (25 marks)
Consider the set $G$ whose elements are the complex $2 \times 2$ matrices $M$ that obey
\[ M^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]
Show that $G$ is a group, with matrix multiplication as the group composition. Then define $G_+$ as a subset of $G$ such that $\det\{M\} = 1$ for all $M \in G_+$, and show that $G_+$ is a subgroup of $G$. Give an example for a group element $M \in G$ that is not in $G_+$.

Problem 2 (25 marks)
The elements of $G_+$ are the matrices $M(a, b, c, d) = \begin{pmatrix} a & ib \\ ic & d \end{pmatrix}$ with the real parameters $a, b, c, d$ subject to $ad + bc = 1$. Which of the restrictions
(a) $b = c$ and $a = d$, (b) $b = 0$, (c) $c = 0$, (d) $M^\dagger = M$
defines a subgroup? Are there Abelian subgroups among them?

Problem 3 (25 marks)
Use Laplace-transformation techniques to evaluate
\[ \int_0^t d\tau (t - \tau)^m \tau^n \]
for $m, n = 0, 1, 2, 3, \ldots$.

Problem 4 (25 marks)
The given hermitian $n \times n$ matrix $S$ is known to be a bit larger than the $n \times n$ identity matrix $E$, in the sense that their difference $S - E$ has small positive eigenvalues. We want to calculate a $n \times n$ matrix $T$ such that $T^\dagger ST = E$ by an iteration of the form
\[ T_0 = E, \quad T_{k+1} = T_k + \lambda T_k (T_k^\dagger ST_k - E) \quad \text{for } k = 0, 1, 2, \ldots, \]
where $\lambda$ is a complex parameter that we wish to choose optimally. Determine the best choice for $\lambda$ by the following strategy. First express $\epsilon_{k+1} = T_{k+1}^\dagger ST_{k+1} - E$ as a cubic polynomial in $\epsilon_k = T_k^\dagger ST_k - E$, that is
\[ \epsilon_{k+1} = c_1 \epsilon_k + c_2 \epsilon_k^2 + c_3 \epsilon_k^3, \]
and then choose $\lambda$ such that $c_1$ vanishes and $|c_2|$ is as small as possible. For your choice of $\lambda$, now find $\epsilon_0$, $\epsilon_1$, and $\epsilon_2$. — Express $S^{-1}$ in terms of $T$. 