1. **Quantum kinematics.** Consider motion along the $x$ axis with, as usual, position operator $X$ and momentum operator $P$ as well as their respective eigenkets $|x\rangle$ and $|p\rangle$.

(a) Show that the operator defined by

$$U = \int_{-\infty}^{\infty} d\lambda \, |p = \lambda p_0\rangle \sqrt{p_0 x_0} \langle x = \lambda x_0|$$

is unitary, whereby the parameter $x_0 > 0$ is a reference length and $p_0 > 0$ is a reference momentum. [8 marks]

(b) Show that $U$ turns position kets into momentum kets in accordance with

$$U|x\rangle = |p = p_0 x/x_0\rangle \sqrt{p_0/x_0}.$$

[6 marks]

(c) Conversely, do you get a position bra if $U$ is applied to $\langle p|$? Justify your answer. [3 marks]

(d) Show that

$$U^\dagger f(X, P)U = f(-x_0 P/p_0, p_0 X/x_0)$$

for any function of $X$ and $P$. [8 marks]

2. **Temporal evolution.** The Hamilton operator of a two-dimensional system is

$$H = \frac{1}{2M} (P_1^2 + P_2^2) + \omega (X_1 P_2 - X_2 P_1)$$

with constant mass $M$ and frequency $\omega$, and $[X_j, P_k] = i\hbar \delta_{jk}$ for $j, k = 1, 2$.

(a) State the Heisenberg equations of motion for $X_1$, $X_2$, $P_1$, and $P_2$. [4 marks]

(b) Show that $P_1^2 + P_2^2$ and $X_1 P_2 - X_2 P_1$ are constants of motion. [6 marks]

(c) Solve the equations of motion of part (a) for $P_1(t)$ and $P_2(t)$. [7 marks]

(d) Determine the time transformation function $\langle x_1, x_2, t|p_1, p_2, t_0\rangle$. [8 marks]
3. Orbital angular momentum. As usual we denote by $L_1$, $L_2$, and $L_3$ the cartesian components of the orbital angular momentum vector operator $\vec{L}$, and the common eigenkets of $\vec{L}^2$ and $L_3$ by $|l, m\rangle$. The state of the system is given by a ket of the form

$$|\rangle = |l = 1, m = 1\rangle \alpha + |l = 1, m = -1\rangle \beta,$$

where $\alpha$ and $\beta$ are complex coefficients with $|\alpha|^2 + |\beta|^2 = 1$.

(a) Determine the expectation values of $L_1$, $L_2$, and $L_3$ as well as their spreads $\delta L_1$, $\delta L_2$, and $\delta L_3$. [10 marks]

(b) For each pair of the spreads $\delta L_1$, $\delta L_2$, and $\delta L_3$, state the uncertainty relation that the pair obeys. [5 marks]

(c) Verify that the equal sign applies in the uncertainty relation for $\delta L_1$ and $\delta L_2$ if $\alpha = \frac{3}{5}$ and $\beta = \frac{4}{5}$. [5 marks]

(d) What can you say, quite generally, about the coefficients $\alpha$ and $\beta$ if the equal sign applies in the uncertainty relation for $\delta L_1$ and $\delta L_2$? [5 marks]

4. Perturbed oscillator. A harmonic oscillator (ladder operators $A$, $A^\dagger$; circular frequency $\omega$) is perturbed by a cubic interaction of strength $\hbar \Omega$,

$$H = H_0 + H_1 \quad \text{with} \quad H_0 = \hbar \omega A A^\dagger, \quad H_1 = \hbar \Omega (A^\dagger A A^\dagger + A A^\dagger A).$$

(a) Determine the 1st-order change of the $n$th energy level. [5 marks]

(b) Determine the 2nd-order change of the $n$th energy level. [10 marks]

(c) What is the energy spacing $\Delta E_n$ between the $n$th and the $(n - 1)$th level when the perturbation $H_1$ is taken into account up to 2nd order? [5 marks]

(d) For which values of $n$ will it surely be necessary to include higher than 2nd-order corrections? [5 marks]