1. Scattering in one dimension. A particle of mass $M$ moves along the $x$-axis with energy $E = \frac{(\hbar k)^2}{2M} \ (k > 0)$ and is scattered by the double delta potential

$$V(x) = -\frac{\hbar^2}{Ma} \delta(x - L/2) - \frac{\hbar^2}{Ma} \delta(x + L/2).$$

The length parameter $a$ determines the strength of the potential, with $a > 0$ for an attractive potential and $a < 0$ for a repulsive potential. As usual, denote the wave function for the given $k$ value by $\phi(x)$, and decompose $\phi(x)$ into the right-moving part $\phi_+(x)$ and the left-moving part $\phi_-(x)$.

(a) Show that the action of the individual delta potentials can be summarized by

$$\begin{pmatrix} \phi_+(0) \\ \phi_-(0) \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \phi_+(−L) \\ \phi_−(0) \end{pmatrix}$$

and

$$\begin{pmatrix} \phi_+(L) \\ \phi_−(0) \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \phi_+(0) \\ \phi_−(L) \end{pmatrix}$$

with the transmission amplitude $t = e^{i(\alpha + \beta) \cos \beta}$ and the reflection amplitude $r = ie^{i(\alpha + \beta) \sin \beta}$, where $\alpha = kL$ and $\cot \beta = ka$. [10 marks]

(b) Determine the $2 \times 2$ scattering matrix for the total scattering potential $V(x)$, that is: find the transmission coefficient $R$ and the reflection coefficient $T$ in

$$\begin{pmatrix} \phi_+(L) \\ \phi_−(−L) \end{pmatrix} = \begin{pmatrix} T & R \\ R & T \end{pmatrix} \begin{pmatrix} \phi_+(−L) \\ \phi_−(L) \end{pmatrix}$$

in terms of $t$ and $r$. [10 marks]

(c) Which relation must be obeyed by $ka$ and $kL$ so that the reflection probability $|R|^2$ vanishes? [5 marks]

2. Scattering in three dimensions. A particle of mass $M$ and wave vector $\vec{k}$ is scattered by a double Yukawa potential

$$V(\vec{r}) = Y(\vec{r} - \vec{a}) + Y(\vec{r} + \vec{a}) \quad \text{with} \quad Y(\vec{r}) = \frac{V_0}{\kappa r} e^{-\kappa r}$$

where $\kappa > 0$ and $V_0 \neq 0$, and $\vec{a}$ is parallel to $\vec{k}$, that is: $\vec{k} \cdot \vec{a} = ka > 0$.

(a) Find the scattering amplitude $f(\theta)$ and the differential cross section $\frac{d\sigma}{d\Omega}(\theta)$ in Born approximation. [12 marks]

(b) It is observed that no scattering occurs in the three directions for which the scattering angle $\theta$ is such that $\cos \theta = 0$ or $\cos \theta = \pm 2/3$. How big is the spacing $a$ between the scattering centers in terms of the de Broglie wavelength $\lambda = 2\pi/k$? [13 marks]
3. Time-dependent interaction. A two-level atom (ground state ket $|g\rangle$, excited state ket $|e\rangle$, energy spacing $\hbar \omega > 0$, transition operator $\sigma = |g\rangle\langle e|$) is resonant with a single photon mode (harmonic-oscillator ladder operators $A$, $A^\dagger$), to which it couples by the time-dependent Rabi frequency $\Omega(t)$. The dynamics is governed by the Hamilton operator $H(t) = H_0 + H_1(t)$ with

$$H_0 = \hbar \omega (\sigma^\dagger \sigma + A^\dagger A) \quad \text{and} \quad H_1(t) = -\hbar \Omega(t)(A^\dagger \sigma + \sigma^\dagger A),$$

where

$$\Omega(t) = \begin{cases} 
2\pi t/T^2 & \text{for } 0 < t < T/2, \\
2\pi (T - t)/T^2 & \text{for } T/2 < t < T, \\
0 & \text{for } t < 0 \text{ and } t > T.
\end{cases}$$

(a) Show first that $(A^\dagger \sigma + \sigma^\dagger A)^2 = \sigma^\dagger \sigma + A^\dagger A$, and then evaluate the commutator $[H(t_1), H(t_2)]$. [8 marks]

(b) Denote by $\alpha(t)$ the probability amplitude for “atom excited and no photons at time $t$” and by $\beta(t)$ the probability amplitude for “atom in the ground state and one photon at time $t$” and state the coupled Schrödinger equations that they obey. [8 marks]

(c) Solve these Schrödinger equations to answer this question: If at time $t = 0$ the atom is excited and there is no photon, what is the probability that the atom is de-excited and one photon present at time $t = T$? [9 marks]

4. Indistinguishable particles. There are two electrons, one has spin up in the $z$ direction and the spatial wave function $\psi_1(\vec{r}) = \langle \vec{r}|1 \rangle$, the other has spin down in the $z$ direction and the spatial wave function $\psi_2(\vec{r}) = \langle \vec{r}|2 \rangle$. Hereby, $\langle 1|1 \rangle = 1 = \langle 2|2 \rangle$, while $\gamma = \langle 1|2 \rangle$ is arbitrary.

(a) Determine the spatial two-electron wave functions $\psi_s(\vec{r}_1, \vec{r}_2)$ and $\psi_t(\vec{r}_1, \vec{r}_2)$ for the singlet and triplet components, respectively. [9 marks]

(b) State the probabilities for finding the electron pair in the singlet and triplet sector. Express your answers in terms of $\gamma$. [6 marks]

(c) Consider all possible spin states of three electrons. How many spin states are there all together? How many spin states belong to total spin $\frac{3}{2}$, how many to total spin $\frac{1}{2}$? [6 marks]

(d) What is the corresponding situation for the spin states of four electrons? [4 marks]