Problem 1 (20=10+10 points)
The state of a particle of mass $M$ is described by the wave function

$$\psi(\vec{r}, t) = \left(c_0 + \vec{c} \cdot \vec{r} \ e^{-i\omega t}\right) e^{-\frac{1}{2}\kappa^2 r^2} ,$$

where $c_0$ and $\vec{c}$ as well as $\kappa$ and $\omega$ are real parameters.

(a) Find the probability density $\rho(\vec{r}, t)$ and the probability current density $\vec{j}(\vec{r}, t)$.

(b) Which relation between $\kappa$ and $\omega$ follows from the continuity equation obeyed by $\rho(\vec{r}, t)$ and $\vec{j}(\vec{r}, t)$?

Problem 2 (30=6+6+10+8 points)
Consider scattering in one dimension (see pages 101–106 of the notes) by the delta potential

$$V(x) = -\frac{\hbar^2}{Ma} \delta(x - L/2) ,$$

where $a$ is a length parameter with $a > 0$ (attractive potential) or $a < 0$ (repulsive potential).

(a) Explain why $\phi(k, x)$ is of the form

$$\phi(k, x) = \begin{cases} 
\frac{1}{\sqrt{k}} \left[ \phi_+(k, 0) e^{ikx} + \phi_-(k, 0) e^{-ikx} \right] & \text{for } x \leq L/2 , \\
\frac{1}{\sqrt{k}} \left[ \phi_+(k, L) e^{ik(x - L)} + \phi_-(k, L) e^{-ik(x - L)} \right] & \text{for } x \geq L/2 .
\end{cases}$$

(b) Which relation among the “in” amplitudes $\phi_+(k, 0)$, $\phi_-(k, L)$ and the “out” amplitudes $\phi_+(k, L)$, $\phi_-(k, 0)$ follows from the continuity of $\phi(k, x)$ at $x = L/2$?

(c) Why is the derivative of $\phi(k, x)$ discontinuous at $x = L/2$ as stated by

$$\left. \frac{\partial \phi(k, x)}{\partial x} \right|_{x = L/2+0} - \left. \frac{\partial \phi(k, x)}{\partial x} \right|_{x = L/2-0} = -\frac{2}{a} \phi(k, x = L/2) ?$$

Use this to find a second relation among the “in” and “out” amplitudes.

(d) Now establish $\alpha = kL$ for the phase on pages 105/106, and then get the scattering matrix $S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}$ from the relations found in parts (b) and (c). Express the matrix elements of $S$ in terms of $k$ and $a$. 
**Problem 3** (25=10+9+6 points)
A two-level atom with unperturbed Hamilton operator $H_0 = \hbar\omega\sigma^\dagger\sigma$ (see page 65 of the notes) is exposed to a time-independent perturbation that is specified by

$$H_1 = \hbar\Omega(\sigma^\dagger + \sigma) \quad \text{with} \quad \Omega > 0.$$ 

At the initial time, the atom is in the ground state $|g\rangle$ of $H_0$.

(a) For short times, the probability $\text{prob}(g \to g, t)$ for remaining in the ground state of $H_0$ is of the form $\text{prob}(g \to g, t) = 1 - (\gamma t)^2$. Determine the value of $\gamma$.

(b) Express $\mathcal{H}_1(t) = e^{iH_0t/\hbar}H_1e^{-iH_0t/\hbar}$ as a linear combination of $\sigma$, $\sigma^\dagger$, $\sigma^\dagger\sigma$, and $\sigma\sigma^\dagger$.

(c) What is the probability, to lowest order in $\Omega$, for finding the atom in the excited state $|e\rangle$ of $H_0$ after time $T$ has elapsed?

**Problem 4** (25=12+3+10 points)
In the Born approximation (see page 125 of the notes), a certain scattering potential $V(\vec{r})$, which is centered at $\vec{r} = 0$, has the scattering amplitude $f(\vec{k}', \vec{k})$ and the differential cross section $d\sigma d\Omega$.

(a) What are the scattering amplitude $f_+(\vec{k}', \vec{k})$ and differential cross section $d\sigma_+ d\Omega$ for the potential $V_+(\vec{r}) = V(\vec{r} - \vec{a})$, centered at $\vec{r} = \vec{a}$?

(b) What are the corresponding $f_-(\vec{k}', \vec{k})$ and differential cross section $d\sigma_- d\Omega$ for $V_-(\vec{r}) = V(\vec{r} + \vec{a})$, centered at $\vec{r} = -\vec{a}$?

(c) Now determine the differential cross section $d\sigma_2 d\Omega$ for the two-center scattering potential $V_2(\vec{r}) = V_+(\vec{r}) + V_-(\vec{r})$. 